University of Texas at Dallas

QUALIFYING EXAM

Real Analysis	
Exam Date:	**Identification Number:**
August 4, 2023	....

Instructions:

1. Use the space provided to write your solutions in this booklet to write the final (neat, elegant and precise) version of your solutions.
3. Provide all the necessary definitions and state all the theorems that you need for your solutions.

Question	**Weight**	**Your Score**	**Comments**
1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

Problem 1.. Assume that $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}$ a differentiable function. Show that for every $k = 1, 2, \ldots, n$, the partial derivative

$$\frac{\partial f}{\partial x_k} : U \to \mathbb{R},$$

is $\mathscr{B}_n$-measurable (here $\mathscr{B}_n$ stands for the σ-algebra of Borel sets in $\mathbb{R}^n$).

SOLUTION:

Problem 2. Assume that $(X, \mathscr{S}_X, \mu_X)$ and $(Y, \mathscr{S}_Y, \mu_Y)$ be two measure spaces. Define the function $\nu^* : \mathscr{P}(X \times Y) \to \overline{\mathbb{R}}$ by

$$\nu^*(C) := \inf \left\{ \sum_{k=1}^{\infty} \mu_X(A_k)\mu_Y(B_k) : C \subset \bigcup_{k=1}^{\infty} A_k \times B_k,\ A_k \in \mathscr{S}_X,\ B_k \in \mathscr{S}_Y \right\}$$

Check if the function ν^* is an outer measure on $X \times Y$.

SOLUTION:

Problem 3. Let $(X, \mathscr{S}, \mu)$ be a measure space, $E \in \mathscr{S}$. Show that if a function $f : E \to \overline{\mathbb{R}}$ is summable then

$$\mu\{x \in E : |f(x)| = \infty\} = 0.$$

SOLUTION:

Problem 4. Show that

(a) if $F : X \to Y$ is a function and $\mathscr{C}$ a σ-algebra in Y then

$$F^{-1}(\mathscr{C}) := \{F^{-1}(B) : B \in \mathscr{C}\}$$

is a σ-algebra in X.

(b) if $\mathscr{K}$ is a class of sets in Y then

$$\mathscr{S}(F^{-1}(\mathscr{K})) = F^{-1}(\mathscr{S}(\mathscr{K})),$$

where $\mathscr{S}(\mathscr{L})$ denotes the smallest σ-algebra containing $\mathscr{L}$.

SOLUTION:

University of Texas at Dallas

QUALIFYING EXAM

Functional Analysis	
Exam Date:	**Identification Number:**
AUGUST 4, 2023	....

Instructions:

1. Use the space provided to write your solutions in this booklet to write the final (neat, elegant and precise) version of your solutions.
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Question	**Weight**	**Your Score**	**Comments**
1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

Problem 1.. Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space and $\mathbb{L}$ a closed subspace. Denote by $\mathbb{E}/\mathbb{L} := \{x + \mathbb{L} : x \in \mathbb{E}\}$ the *quotient space space* of all $\mathbb{L}$-cosets $x + \mathbb{L}$ in $\mathbb{E}$. We define the function $\|\cdot\|_o : \mathbb{E}/\mathbb{L} \to \mathbb{R}$ by

$$\|x + \mathbb{L}\|_o := \inf_{y \in \mathbb{L}} \|x - y\| =: \operatorname{dist}(x, \mathbb{L}).$$

(a) Show that $\|\cdot\|_o$ is a norm.

(b) Show that the space $\mathbb{E}/\mathbb{L}$ equipped with the norm $\|\cdot\|_o$ is a Banach space.

SOLUTION:

Problem 2. Let $\mathbb{E}$ be a normed space and $C \subset \mathbb{E}$ a convex open set such that $0 \in C$ Consider the gauge function

$$\forall_{x \in \mathbb{E}} \quad p_C(x) := \inf\{\alpha > 0 : x \in \alpha C\}.$$

(a) Show that p_C is a convex function.

(b) Show that $p_C(x + y) \leq p_C(x) + p_C(y)$ for all $x, y \in \mathbb{E}$.

SOLUTION:

Problem 3. Let $\mathbb{E}$ be a Banach space and $f \in \mathbb{E}^*$, $f \neq 0$. Show that

$$\operatorname{dist}(x, \operatorname{Ker}(f)) = \frac{|f(x)|}{\|f\|}.$$

SOLUTION:

Problem 4: Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ be two Banach spaces and $Y \subset \mathbb{E}$ be a set such that

$$\overline{\operatorname{conv}(Y)} = \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \leq 1\}.$$

Show that for any $A \in L(\mathbb{E}, \mathbb{F})$ we have

$$\|A\| = \sup_{y \in Y} \|Ay\|_{\mathbb{F}}.$$

SOLUTION:

Complex Variables Qualifying Exam

Your work should be turned in at the end of the exam. You should solve the exercises by yourself. You should provide complete solutions. For incomplete solutions, you may receive partial credit or zero credit. This is a closed book exam. You are not permitted to use notes, equation sheets, books, or any other aids. In particular, you are not permitted to use any electronic devices. Your mobile phone must be switched off and should remain in your bag during the entire exam period. Please use blue or black pen. The maximum number of points for each exercise is indicated. ***The exam has a total of 60 points.***

1. **True** or **False** (Justification is required – provide a proof or a counterexample)
   (a) If $u(x, y)$ and $v(x, y)$ are harmonic in a domain $D \subset \mathbb{C}$, then the product $u(x, y) \cdot v(x, y)$ is harmonic on D. [4]
   (b) There exist two distinct harmonic functions on $\mathbb{C}$ that vanish on the entire real axis. [4]
   (c) There exist two distinct analytic functions on $\mathbb{C}$ that vanish on the entire real axis. [4]

2. (a) State Rouche's theorem. [3]
   (b) If $f(z) = 2z^5 + 7z^3 + z^2 - 3$, find the number of zeros of $f(z)$ inside the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. [5]

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function and set $g(z) := f\left(\frac{1}{z}\right)$. Prove that f is a polynomial if and only if $g(z)$ has a pole at $z = 0$. [8]

4. (a) Find the fractional linear transformation $w = f(z)$ for which $f(0) = 0$, $f(2) = 4$, and $f(i) = 1 - i$. Find its inverse. [5]
   (b) Clearly $z = 0$ is a fixed point of f. Are there any others? [2]
   (c) Find the image of the region $\{z = x + iy : 1 \leq y\}$ under f in the w-plane. [5]

5. Use the Calculus of Residues to compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$. [10]

6. Let

$$f(0) = 0 \qquad \text{and} \qquad f(x+iy) = u(x,y) + iv(x,y) \qquad \text{for} \qquad z = x + iy \neq 0,$$

where

$$u(x,\, y) = \frac{x^3 - y^3}{x^2 + y^2}, \qquad v(x,\, y) = \frac{x^3 + y^3}{x^2 + y^2}.$$

(a) Are the Cauchy–Riemann equations satisfied at $z = 0$? [5]

(b) Does the complex derivative $f'(0)$ exist? [5]

Algebra Qualifying Exam

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. Show your work and justify all statements.

Problem 1 (25 points). Let R be a commutative ring with unity.

(a) (10 points) Let M be an ideal of R. Show that R/M is a field iff M is a maximal ideal.

(b) (15 points) Suppose every element $x \in R$ satisfies $x^{m_x} = x$ for some integer $m_x > 1$ (this integer may depend on x). Show that every prime ideal in R is maximal.

Problem 2 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements x, y, z subject to the relations

$$\begin{aligned} -10x + 18y + 40z &= 0 \\ -8x + 8y + 20z &= 0 \\ -16x + 24y + 56z &= 0. \end{aligned}$$

Problem 3 (25 points). Let $\mathbb{F} = \mathbb{Z}_3$, $G = \mathrm{GL}_2(\mathbb{F})$ the group of invertible matrices with entries in $\mathbb{F}$.

(a) (10 points) Compute the order of G using the orbit-stabilizer theorem and the action of G on the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{F}^2$.

(b) (5 points) Let N be the center of G. Find, with proof, the matrices in N.

(c) (10 points) Prove that $G/N \equiv S_4$ by defining an action of G on the four one-dimensional subspaces of $\mathbb{F}^2$

Problem 4 (25 points). Let G be the group of matrices

$$\left\{ \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} : a \in \mathbb{Z}_7, b \in \mathbb{Z}_7^\times \right\}.$$

(a) (5 points) Check that G is a group.

(b) (10 points) Find a Sylow 7-subgroup of G and $n_7(G)$.

(c) (10 points) Find a Sylow 3-subroup of G and $n_3(G)$.

Qualifying Exam. ODE

This is a closed book, closed notes exam. To receive full credit it is sufficient to provide correct solutions to any four out of the proposed five problems.

QE ID:	
Score:	

1	2	3	4	5	$\sum$

Problem 1 (20 pts)**.** How many solutions satisfying the initial conditions $x(0) = 1, x'(0) = 2$ there are for the equation $x^{(n)} = t + x^2$? Give a complete answer with justification for all values of $n \in \mathbb{N}$.

Problem 2 (20 pts)**.** For the differential equation:

$$x^2(x+1)y'' - 2y = 0$$

The function $y_1(x) = \dfrac{x+1}{x}$ is a solution. Find general solution for this equation.

Problem 3 (20 pts)**.** Solve the system of differential equations

$$\begin{cases} \ddot{x} = 2y \\ \ddot{y} = -2x \end{cases}$$

Problem 4 (20 pts)**.** Determine whether the zero solution for the system

$$\begin{cases} \dot{x} = x^2 + y^2 - 2x \\ \dot{y} = 3x^2 - x + 3y \end{cases}$$

is stable or not.

Problem 5 (20 pts)**.** Find all the critical points of the system and determine its stability.

$$\begin{cases} \dot{x} = (2x - y)(x - 2) \\ \dot{y} = xy - 2 \end{cases}$$

Differential Geometry Qualifying Exam

Your work should be turned in at the end of the exam. You should solve the exercises by yourself. You should provide complete solutions. For incomplete solutions, you may receive partial credit or zero credit. This is a closed book exam. You are not permitted to use notes, equation sheets, books, or any other aids. In particular, you are not permitted to use any electronic devices. Your mobile phone must be switched off and should remain in your bag during the entire exam period. Please use blue or black pen. The maximum number of points for each exercise is indicated. ***The exam has a total of 50 points.***

1. Let E be a finite dimensional vector space of dimension n. A p-form $\alpha \in \Lambda^p E^*$ is said to be **decomposable** if it can be written as the wedge product of p 1-forms, i.e., if $\alpha = \beta_1 \wedge \cdots \wedge \beta_p$ for $\beta_i \in \Lambda^1 E^*$, where $i = 1, \ldots, p$.
   (a) Show that every element of $(\Lambda^n E^*) \setminus \{0\}$ is decomposable. [4]
   (b) Let $\alpha \in \left(\Lambda^1 E^*\right) \setminus \{0\}$. Show that $\omega \in \Lambda^p E^*$ satisfies $\omega = \alpha \wedge \omega'$ for some $\omega' \in \Lambda^{p-1} E^*$ if and only if $\alpha \wedge \omega = 0$. [6]

2. (a) State Stokes's theorem. [3]
   (b) Let M be a compact, connected, orientable smooth manifold of dimension 6 without boundary. Let α and β be two 2-forms on M.
      (i) Show that $d\alpha \wedge d\beta$ is exact. [2]
      (ii) Show that there is a point of M where $d\alpha \wedge d\beta = 0$. [5]

3. Consider the unit 2-sphere $S^2 \subset \mathbb{R}^3$, defined by the equation $x^2 + y^2 + z^2 = 1$. Let $\omega = z\, dx \wedge dy$. Compute $\int_{S^2} \omega|_{S^2}$. [10]

   Hint: $S^2 \subset \mathbb{R}^3$ can be parametrized by $g : [0, \pi] \times [0, 2\pi) \to S^2$:

   $$g(\theta, \varphi) = (x, y, z) = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)), \qquad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi).$$

4. Consider the map $f : \mathbb{R}P^2 \to \mathbb{R}^3$ defined by

$$f([x:y:z]) = \left(\frac{yz}{x^2+y^2+z^2}, \frac{zx}{x^2+y^2+z^2}, \frac{xy}{x^2+y^2+z^2} \right).$$

(a) Show that f is well-defined, i.e., $f([\lambda x : \lambda y : \lambda z]) = f([x:y:z])$ for all $\lambda \neq 0$. [3]

(b) Show that f is a smooth map. [7]

5. Let (u, v) and (x, y, z) denote coordinates on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, and consider the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (u, v) = (x^2 + y^2, yz)$.

(a) Calculate $f^*(udv + vdu)$. [5]

(b) Calculate $f_* \left(\left. \frac{\partial}{\partial y} \right|_{(1,3,-1)} \right)$. [5]

MATH 6319 - Principles and Techniques of Applied Mathematics
Ph. D Qualifying Examinations
Summer 2023
Felipe Pereira & V. Ramakrishna

REQUIRED: Please enter the unique ID created for you for the QEs *on this line:*

It is recommended that you enter the ID, from the previous line, on each page of your answers.

CAUTION: Do **NOT** enter your name or UTD ID anywhere.

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** Let $(X, <, >)$ be a pre-inner product space. Show that the Gram matrix of a finite subset of X is psd. Now let $(X, <, >)$ be an inner-product space. Show then that a Gram matrix of a finite set is positive definite iff this set is linearly independent. What parts of this go through if the the non-degeneracy axiom of an inner product does not go through?

  Use Gram matrices to prove the CBS inequality.

- **Q2** State the Crabtree-Haynsworth formula. Derive it by using a result on the Schur complement in a threefold product of matrices.

- **Q3** Define the SVD of a rectangular matrix. Let $A_{n\times n}$ be non-singular. Find the closest singular matrix to A in the spectral norm, and justify your answer.

- **Q4** Let V be a complex pre-inner product space. Find a suitable quotient space V/W and suitable inner-product on it which is related to that on V. Justify your answer fully (i.e., identify W, explain why it is a subspace; explain why your inner product is well defined and why it is indeed an inner product etc.,) You may assume the contents of **Q1**, but you must state them at the precise point of usage.

Q5 How can the Fourier transform be extended to L^2? You must state precisely any density reult that you use. State the Plancherel-Parseval theorem.

(13 + 12 = 25 points)

Q6 Show that the Fourier transform of a Gaussian is also a Gaussian. All statements about differentiation and the Fourier transform must be stated precisely at the point of usage.

MATH 6322
Mathematical Foundations of Data Science
Ph. D Qualifying Examinations
Summer 2023
V. Ramakrishna & N. Wu

REQUIRED: Please enter the unique ID created for you for the QEs *on this line:*

It is recommended that you enter the ID, from the previous line, on each page of your answers.

CAUTION: Do **NOT** enter your name or UTD ID anywhere.

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** Consider data $\{x_i, i = 1, \ldots, N\} \subseteq R^m$ which is to be clustered into K clusters, $\Gamma_a, a = 1, \ldots, K$. Define vectors $e_a \in R^n, a = 1, \ldots, K$ via $(e_a)_l = 1$ if x_l is in the ath cluster and 0 otherwise. Here $(e_a)_l, l = 1, \ldots, N$ is the lth component of the vector e_a. Next define a square $N \times N$ matrix $Z = \Sigma_{a=1}^{K} \frac{1}{|\Gamma_a|} e_a e_a^T$.

  - i) Explain why Z is positive semidefinite.
  - ii) Explain why Z is entrywise non-negative.
  - iii) Show $\mathrm{T}r(Z) = K$.
  - iv) Show $Ze = e$, where e is the vector of all ones in R^N.
  - v) Define a square $N \times N$ matrix D, concocted out the data set, such that the K-means cost function reduces to $\frac{1}{2}\mathrm{T}r(Z^T D)$.
  - vi) State what the Peng-Wei SDP relaxation of K-means is.

  (4 + 4 + 4 + 4 + 4 + 5 = 25 points)

- **Q2**

- i) Explain why the product of two positive definite kernels is also one.
- ii) What is one feature map which yields the inner product on a Hilbert Space as a kernel?
- iii) Define the Gaussian kernel.
- iv) Prove that the Gaussian kernel is indeed a kernel. All auxilliary results must be stated precisely at the point of usage.

$(7 + 7 + 5 + 6 = 25$ points)

- **Q3** Explain how $max(x_1, x_2)$ can be implemented exactly by a feedforward neural network with ReLU activation. What advantage is achieved if $x_2 \geq 0$ in terms of the archirecture of the network?

(25 points)

- **Q4** State and prove a sufficient condition for the decomposability of the nuclear norm of a matrix.

(25 points)

Q5 Define i) non-negative rank, $r_+(A)$; ii) the semi non-negative rank, $r_S(X)$ and iii) Coherence of a matrix and state and derive the Welch bound for it.

$(6 + 6 + 13 = 25$ points)

Q6 In this question you will prove that a finite hypothesis class is agnostic PAC learnable. Do it via the following steps:

- Show that the true risk, $L_P(h)$, is the expected value of the empirical risk $L_S(h)$.
- If a training set is $\epsilon/2$ representative then for any output h_S of the ERM algorithm we have $L_P(h_S) \leq min_{\tilde{h}\in H} L_P(\tilde{h}) + \epsilon$.

- Show, using appropriate large deviation inequalities, that if H is a finite hypothesis class and l is a loss-function with values in $[0, 1]$, then H has the uniform convergence property and therefore that it is agnostic PAC learnable.

$(7 + 7 + 11 = 25$ points)

Numerical Analysis Qualifying Exam August 2023

QE ID:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please work all 4 of these problems.

(1) Derive a formula for the error in the approximation

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)].$$

(2a) Derive a formula for approximating

$$\int_1^3 f(x)dx$$

in terms of $f(0)$, $f(2)$, $f(4)$. It should be exact for all f in Π_2.

(b) If the formula

$$\int_1^2 (x^4 - 1)f(x)dx = Af(x_0) + Bf(x_1) + Cf(x_2)$$

is correct for all f that are polynomials of degree ≤ 5, then x_0, x_1, and x_2 must be roots of a polynomial q having what properties?

(3a) Give the Taylor series derivation of the scalar version of Newton's Method and from this derivation show that Newton's Method is quadratically convergent.

(b) What is the main condition required for Newton's Method to converge?

(c) Will either of the following iterations converge to the indicated fixed point α? If the iteration does converge, does it converge quickly or slowly to the fixed point? Justify your answer.

(a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, \quad \alpha = 2$
(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, \quad \alpha = 3^{1/3}$

(4) Is 1/9 a machine number on the Marc 32? If 1/9 is not a machine number on the Marc 32, determine the binary machine number just to the right of 1/9 on the Marc 32 machine. What is the exact roundoff error in this approximation? (Hint: in 1+ form, the Marc 32 has a 24 bit mantissa.)

Numerical PDE Qualifying Exam August 2023

QE ID:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please work all 4 of these problems.

(1) Consider the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0$$

with initial conditions $u^1(0,x) = 0$ and $u^2(0,x) = x$.

(a) What are the characteristic curves for this problem?

(b) How many boundary conditions must be specified for the solution to be uniquely defined for $t \geq 0$ and $x \in [0,1]$? (Hint: draw a picture).

(2) Consider the one-way wave equation $u_t + au_x = 0$ with $a > 0$. Prove an order of accuracy estimate for the backward-time, backward-space scheme. The scheme is:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^{n+1} - v_{m-1}^{n+1}}{h} = 0.$$

(3a) Consider the 1D heat equation $u_t = bu_{xx}$, with $b > 0$, $b \in \Re$. Here $u(t,x)$ is the temperature at time t, position x. Using the trigonometric relationship $\sin^2(x) = \frac{1-\cos(2x)}{2}$, prove stability for the forward-time, central-space scheme:

$$\frac{v_m^{n+1} - v_m^n}{k} = b\left[\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}\right].$$

(b) If $\mu = \frac{k}{h^2}$, what is the stability requirement on μ?

(4a) For the boundary value problem

$$-\frac{d^2u}{dx^2} = 3 \qquad 0 < x < 1$$

$$u(0) = 0, \qquad u(1) = 0$$

Derive the Galerkin weak formulation of the problem.

(b) Derive the load vector f for this problem using piecewise linear finite elements and a regular mesh of gridpoints $x = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$.

(c) Modify the boundary conditions for this problem to

$$\frac{du}{dx}(0) = 0, \qquad u(1) = 2.$$

Now derive the load vector f for the modified problem. (Note the size of the new load vector for this problem.)