



University of Texas at Dallas

QUALIFYING EXAM

Real Analysis	
Exam Date: January 9, 2023	Identification Number:

Instructions:

1. Use the space provided to write your solutions in this booklet to write the final (neat, elegant and precise) version of your solutions.
2. Provide all the necessary definitions and state all the theorems that you need for your solutions.

Question	Weight	Your Score	Comments
1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

NOTATION:

- We denote by \mathbf{m}_n the *Lebesgue measure* on the space \mathbb{R}^n , which is defined on the σ -algebra \mathcal{L}_n of Lebesgue measurable sets in \mathbb{R}^n .
- We also denote by \mathcal{B}_n the σ -algebra of Borel sets in \mathbb{R}^n .
- The *outer Lebesgue measure* on \mathbb{R}^n is denoted by $\mathbf{m}_n^* : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty]$.
- We say that a bounded set $A \subset \mathbb{R}^n$ is *Jordan measurable* if its *outer Jordan measure* $\nu^*(A)$ and *inner Jordan measure* $\nu_*(A)$ are equal. In such a case we write $\nu(A) = \nu^*(A) = \nu_*(A)$ to denote the *Jordan measure* of A .
- For a measure space (X, \mathcal{S}, μ) we denote by $f_n \xrightarrow{\mu} f$ the *convergence in measure* of a sequence $\{f_n\}$ to f .

Problem 1. Suppose that $A \subset \mathbb{R}$ is a non-measurable set with respect to the Lebesgue measure μ_1 , i.e. $A \notin \mathcal{L}_1$. Define the sets

$$E_1 := \{(x, 0) : x \in A\} \quad \text{and} \quad E_2 := \{(x, y) : x \in A, y \in [0, 1]\}.$$

Verify if

- (a) the sets E_1 and E_2 are \mathfrak{m}_2 -measurable (i.e. if $E_i \in \mathcal{L}_2$, $i=1,2$);
- (a) the sets E_1 and E_2 are Borel sets in \mathbb{R}^2 (i.e. if $E_i \in \mathcal{B}_2$, $i=1,2$);

Justify your answers.

SOLUTION:

Problem 2. Assume that $S \subset \mathbb{R}^n$ is a dense countable set. Put

$$E := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x \notin S, |x_i| \leq 1, i = 1, 2, \dots, n\}$$

- (a) Is E Jordan measurable?
- (b) Is E Lebesgue measurable (if it is, what is $\mathbf{m}_n(E)$)?
- (c) Is E a Borel set?

Justify your answers.

SOLUTION:

Problem 3. Let (X, \mathcal{S}, μ) be a finite measure space (i.e. $\mu(X) < \infty$). Assume that $f_n : X \rightarrow \overline{\mathbb{R}}$ and $g_n : X \rightarrow \overline{\mathbb{R}}$ are two sequences of μ -measurable functions such that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ (for some measurable functions f and $g : X \rightarrow \overline{\mathbb{R}}$). Show that $f_n g_n \xrightarrow{\mu} fg$.

Solution:

Problem 4: Suppose that $A \subset \mathbb{R}^n$. Show that

(a) if $\mathbf{m}_n^*(A) = 0$ then $\text{int}(A) = \emptyset$;

(b) if $\mathbf{m}_n^*(\partial A) = 0$ then A is Lebesgue measurable (i.e. $A \in \mathcal{L}_n$).

SOLUTION:



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QUALIFYING EXAM

Functional Analysis	
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1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

Problem 1. Consider an infinite-dimensional Banach space \mathbb{E} and suppose that $\{f_1, f_2, \dots, f_n\} \subset \mathbb{E}^*$ and $f : \mathbb{E} \rightarrow \mathbb{R}$ a linear functional such that

$$\bigcap_{k=1}^n \ker(f_k) \subset \ker(f).$$

Show that

(a) the subspace

$$\mathbb{L} := \bigcap_{k=1}^n \ker(f_k)$$

is infinite dimensional.

(b) f is continuous;

(c) There exists constants α_k , $k = 1, 2, \dots, n$ such that

$$\forall x \in \mathbb{E} \quad f(x) = \sum_{k=1}^n \alpha_k f_k(x).$$

SOLUTION:

Problem 2. Let \mathbb{E} be a normed space and $f : \mathbb{E} \rightarrow \mathbb{R}$ a non-zero linear functional. Show that the following conditions are equivalent

- (a) f is continuous;
- (b) $\text{Ker}(f)$ is closed;
- (c) $\text{Int}(\overline{\text{Ker}(f)}) = \emptyset$.

SOLUTION:

Problem 3. Assume that $a \in \mathbb{R}^n$ is a fixed vector and $b \in \mathbb{R}$. Define the function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\varphi(x) = a \bullet x + b, \quad x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n,$$

where $x \bullet y := \sum_{k=1}^n x_k y_k$. Compute $\varphi^* : (\mathbb{R}^n)^* \rightarrow (-\infty, \infty]$. What is φ^{**} ?

Solution:

Problem 4: Suppose that \mathbb{E} is a reflexive infinite-dimensional Banach space such that \mathbb{E}^* is separable. Show that there exists a sequence $\{x_n\}$ in \mathbb{E} such that

$$\forall_{n \in \mathbb{N}} \|x_n\| = 1 \quad \text{and} \quad x_n \rightharpoonup 0 \quad \text{as} \quad n \rightarrow \infty.$$

SOLUTION:

Complex Analysis Qualifying Exam

Spring 2023

Friday, January 13, 2023

1. [25 points] True or false (Justification is needed):

(a) The function $f(z) = 2z + \bar{z}^2$ is differentiable for $\forall z \in \mathbb{C}$.

(b) If f is analytic within and on the simple closed contour γ and z_0 is a point within γ , then

$$\int_{\gamma} \frac{f'(z)}{z - z_0} dz = \int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz.$$

(c) The function $f(z) = \frac{\sin(\pi z)}{(z-1)^3}$ has a pole of order three at $z = 1$.

2. [25 points]

(a) State Rouché's Theorem.

(b) Compute the number of solutions, including multiplicity, of the equation

$$z^5 \cos z + 5i z^4 + 2 = 0$$

inside the unit disk $|z| < 1$.

3. [25 points] If $f(z)$ is an entire function satisfying the estimate

$$|f(z)| \leq 1 + |z|^{\sqrt{2023}} \quad \forall z \in \mathbb{C},$$

show that $f(z)$ is a polynomial and determine the best upper bound for the degree of $f(z)$.

4. [25 points] Use the calculus of residues to evaluate the integral

$$\int_0^{+\infty} \frac{x \sin 5x}{x^2 + 4} dx.$$

Verify all steps of the calculation.

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. Show your work and justify all statements.

Problem 1 (25 points).

(a) (15 points) Let G be a group with a normal subgroup H of index n . Show that for any $a \in G$, we have $a^n \in H$.

(b) (10 points) Give an example of a group G with a subgroup H of index n and an element $a \in G$ with $a^n \notin H$.

Problem 2 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements x, y, z subject to the relations

$$-55x + 5y - 30z = 0$$

$$175x - 25y + 100z = 0$$

$$10x + 5z = 0.$$

Problem 3 (25 points).

- (a) (5 points) Let G be a finite group acting on a finite set X . State Burnside's lemma for the number of orbits $|X/G|$.
- (b) (20 points) How many inequivalent ways are there to color each edge of a regular tetrahedron with one of n colors and each vertex with one of m colors? (Two colorings are considered equivalent if there is an element of group of symmetries of the tetrahedron taking one of the colorings to the other.)

Problem 4 (25 points).

- (a) (5 points) Let $X = \text{Syl}_5(S_5)$ be the set of Sylow 5-subgroups of the symmetric group S_5 . List the elements of X . How many are there?
- (b) (10 points) Write S_X for the symmetric group on the set $X = \text{Syl}_5(S_5)$. Prove that the action (by conjugation) of S_5 on X gives a homomorphism $\phi : S_5 \rightarrow S_X$ with trivial kernel.
- (c) (10 points) Let S_X act on the cosets $S_X/\text{im}(\phi)$. Describe the action of a transposition from S_X acting on these cosets.

Qualifying Exam. ODE

This is a closed book, closed notes exam. To receive full credit it is sufficient to provide correct solutions to any four out of the proposed five problems.

QE ID:						
Score:	1	2	3	4	5	Σ

Problem 1 (20 pts). Write down the differential equation on $y(x)$ whose solutions parametrize the set of all parabolas passing through the origin with the axis parallel y -axis.

Problem 2 (20 pts). Solve the differential equation:

$$y' = \frac{1}{x - y^2}.$$

Problem 3 (20 pts). For which values of the parameters α , β and γ functions $\sin(\alpha t)$, $\sin(\beta t)$ and $\sin(\gamma t)$ are linearly independent?

Problem 4 (20 pts). Find the periodic solution of the equation

$$\ddot{x} + a\dot{x} + bx = \sin(\omega t).$$

Problem 5 (20 pts). Solve the system of differential equations

$$\begin{cases} \dot{x} = 2x - y - z \\ \dot{y} = 3x - 2y - 3z \\ \dot{z} = 2z - x + y \end{cases}$$

QE ID:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 4 of the 5 problems to solve. Please indicate which 4 problems you would like graded.

(1) Let $x = 2^{16} + 2^{-8} + 2^{-9} + 2^{-10}$

(a) Find the machine number x^* closest to x in the Marc-32. (Hint: the Marc-32 is a hypothetical 32-bit computer that follows IEEE standards. This machine represents a nonzero real number in binary using 1 bit for the sign of the real number, 8 bits for storing the exponent, and the remaining 23 bits for the mantissa.)

(b) For this number show that the relative error between x and x^* is no greater than the unit roundoff error for the Marc-32.

(2) If r is a zero of multiplicity 2 of the function f (i.e., $f(r) = f'(r) = 0 \neq f''(r)$), then Newton's method exhibits linear convergence. **Prove** that quadratic convergence in Newton's iteration will be restored by making the modification:

$$x_{n+1} = x_n - 2f(x_n)/f'(x_n).$$

(3) Given $f(0)$, $f'(-1)$, and $f''(1)$, compute an approximation to

$$\int_{-1}^1 x^2 f(x) dx$$

by the method of undetermined coefficients. Your formula should be exact for all $f \in \Pi_2$ (i.e., the space of polynomials of degree less than or equal to 2).

(b) Is this formula a Gaussian quadrature formula? Why or why not?

(4a) If we wish to solve the first-order initial value problem

$$x'(t) = f(t, x(t)), \quad x(a) = x_a$$

over the interval $[a, b]$ with step size $h = (b-a)/n$, we can use Euler's method, namely, $x(t+h) = x(t) + hf(t, x(t))$. Derive the modified Euler's method:

$$x(t+h) = x(t) + hf\left(t + \frac{1}{2}h, x(t) + \frac{1}{2}hf(t, x(t))\right)$$

by performing Richardson's extrapolation on Euler's method using step sizes h and $h/2$. Hint: Assume the error term is Ch^2 where $C \in \mathfrak{R}$.

(b) What is the order of accuracy of the scheme above?

(5) Prove the Theorem on Characterizing Best Approximation:

Let G be a subspace in an inner-product space E . For $f \in E$ and $g \in G$, these properties are equivalent:

- (1) g is a best approximation to $f \in G$,
- (2) $f - g \perp G$.

**MATH 6319 - Principles and Techniques of
Applied Mathematics
Ph. D Qualifying Examinations
Spring 2023
Felipe Pereira & V. Ramakrishna**

REQUIRED: Please enter the unique ID created for you for the QEs *on this line*:

It is recommended that you enter the ID, from the previous line, on each page of your answers.

CAUTION: Do **NOT** enter your name or UTD ID anywhere.

CHOICE: Do any 4 of the Qs below. Each **Q** is worth 25 points.

- **Q1** Compute in closed form the eigenvalues of a rank 1 matrix. When precisely is a rank one matrix diagonalizable? When it is, what is one choice of a diagonalizing similarity? State all results you will use precisely and at the point of usage.

- **Q2** State the Crabtree-Haynsworth formula. Derive it by using a result on the Schur complement in a threefold product of matrices.

- **Q3** State the rank-nullity theorem. A consequence of it, colloquially stated, is that a linear $T : V \rightarrow V$ is injective iff it is surjective. Give an example to show that it fails for infinite-dimensional V .

Finally derive the solution to the Lagrange interpolation problem via the rank-nullity theorem, including the fact that the Lagrange polynomials also form a basis for P_n .

(5 + 5 + 15 = 25 points)

- **Q4** Let V be a complex pre-inner product space. Find a suitable quotient space V/W and suitable inner-product on it which is related to that on V . Justify your answer fully (i.e., identify W , explain why

it is a subspace; explain why your inner product is well defined and why it is indeed an inner product etc.,) You may assume the contents of the previous result, but you must state them at the precise point of usage.

Q5 How can the Fourier transform be extended to L^2 ? You must state precisely any density result that you use. State the Plancherel-Parseval theorem.

(13 + 12 = 25 points)

Q6 Show that the Fourier transform of a Gaussian is also a Gaussian. All statements about differentiation and the Fourier transform must be stated precisely at the point of usage.