QE ID: ______________________________

Qualifying Exam, August 2022
Math 6301 Real Analysis

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Instructions: Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.

1. Let $A \subset [0,1]$ be a set and μ be the Lebesgue measure on $[0,1]$.
   (a) Show that if $\mu(\overline{A}) = 0$ then A is Jordan measurable, where $\overline{A}$ denotes the closure of A.
   (b) Show that if $\mu(\partial A) = 0$ then A is Lebesgue measurable, where ∂A denotes the boundary of A.

2. Let f be a real valued function on $\mathbb{R}$.

(a) If f is Lebesgue measurable, is $f^{-1}(y)$ Lebesgue measurable for every $y \in \mathbb{R}$? Justify your answer.

(b) If $f^{-1}(y)$ is Lebesgue measurable for any $y \in \mathbb{R}$, must f be Lebesgue measurable? Justify your answer.

3. Find

$$\lim_{n\to\infty} \int_{[0,1]} \cos(x^n)\, d\mu,$$

where μ stands for the Lebesgue measure in $\mathbb{R}$. Justify your answer.

4. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions. Let μ be the Lebesgue measure on $\mathbb{R}$. Prove that if f_n converges to zero in measure, then

$$\lim_{n\to\infty} \int_{[0,1]} \frac{|f_n|}{1+|f_n|}\, d\mu = 0.$$

University of Texas at Dallas

QUALIFYING EXAM

Functional Analysis	
Exam Date:	**Identification Number:**
August 8, 2022	....

Problem 1. Let C be a convex set in a Banach space $\mathbb{E}$. Show that C is weakly closed (i.e. with respect to the $\sigma(\mathbb{E}, \mathbb{E}^*)$ topology) if and only if it is closed (with respect to the strong topology).

Problem 2. Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces and let $T \in L(\mathbb{E}, \mathbb{F})$ be a surjective operator.

(a) Let M be any set of $\mathbb{E}$. Prove that $T(M)$ is closed in $\mathbb{F}$ if and only if $M + \mathrm{Ker}(T)$ is closed in $\mathbb{E}$. (**Hint:** To show $\Leftarrow$ use the fact that $\widetilde{\mathbb{E}} := \mathbb{E}/\ker(T)$ is a Banach space and the operator T can be factorized by $\widetilde{T} : \widetilde{\mathbb{E}} \to \mathbb{F}$.)

(b) Assume that $L \subset \mathbb{E}$ is a closed linear subspace and $\dim \mathrm{Ker}(T) < \infty$. Show that $T(L)$ is also a closed subspace.

Problem 3. Suppose that $\mathbb{E}$ is a reflexive infinite-dimensional Banach space such that $\mathbb{E}^*$ is separable. Show that there exists a sequence $\{x_n\}$ in $\mathbb{E}$ such that

$$\forall_{n\in\mathbb{N}} \ \ \|x_n\| = 1 \quad \text{and} \quad x_n \rightharpoonup 0 \quad \text{as} \quad n \to \infty.$$

Problem 4. Consider an infinite-dimensional Banach space $\mathbb{E}$ and suppose that $\{f_1, f_2, \ldots, f_n\} \subset \mathbb{E}^*$. Show that the subspace

$$\mathbb{L} := \bigcap_{k=1}^{n} \ker(f_k)$$

is infinite dimensional.

Complex Analysis Qualifying Exam

Summer 2022

Friday, August 12, 2022

1. [**25 points**] **True** or **false** (Justification is needed):

- **(a)** There is a nonconstant entire function such that $|f(z)| \quad 1 \quad \sqrt{|z|}$ for all $z \quad C$.
- **(b)** If □ is a contour, and z_0 a point of $C \setminus \quad$, then there is a disk centered at z_0 such that the winding number function $n(\quad , z)$ is constant for all z from that disk.
- **(c)** If $\quad_{n} a_n z^n$ has radius of convergence *R*, then $\quad_{n} \mathrm{Re}(a_n) z^n$ has radius of convergence greater or equal to *R*.

2. [**25 points**]

Let p t□1 be an integer and f a holomorphic function on the unit disk $D \quad \{z \,|\, |z| \quad 1\}$ such that

- (a) $|f(z)| \quad |z|^p$ for all z with $|z| < 1$;
- (b) f has a zero of order greater or equal to p at 0.

Assume further the existence of a z□0 such that $f(a) \quad a^p$. What can be said of f ?

3. [**25 points**]

Let $f(z)$ be an analytic function on ***C*** which takes value in the upper half plane, i.e., $f: C \quad H$, where $H \quad \{x \quad iy \,|\, y \quad 0\}$. Show that $f(z)$ is constant.

4. [**25 points**] Use the calculus of residues to evaluate the integral

$$\frac{dx}{(x^2 \quad 4)^2 (x^2 \quad 16)}.$$

Verify all steps of the calculation.

Algebra Qualifying Exam

Fall 2022

QE ID:

Instructions. Write your QE ID above and solve all four problems. Show your work and justify all statements.

Problem 1 (25 points). Let G be a group and consider the set

$$H = \{(g, g) : g \in G\} \subseteq G \times G.$$

(a) (10 points) Show that H is a subgroup of $G \times G$ isomorphic to G.

(b) (15 points) Show that H is normal in $G \times G$ if and only if G is abelian.

Problem 2 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements x, y, z subject to the relations

$$\begin{aligned} 18x - 6y + 15z &= 0 \\ -39x + 9y - 45z &= 0 \\ -243x + 99y - 117z &= 0. \end{aligned}$$

Problem 3 (25 points).

(a) (5 points) Let G be a finite group acting on a finite set X. State Burnside's lemma for the number of orbits $|X/G|$.

(b) (20 points) Let n and k be positive integers. Find a formula the number of ways to simultaneously color each edge of a square using one of n colors and each vertex with one of k colors, where two colorings are considered equivalent if there is an element of the dihedral group of symmetries of a square taking one of the colorings to the other.

Problem 4 (25 points).

(a) (5 points) Describe the set $X = \mathrm{Syl}_5(S_5)$ of Sylow 5-subgroups of the symmetric group S_5.

(b) (10 points) Write S_X for the symmetric group on the set $X = \mathrm{Syl}_5(S_5)$. Prove that the action of S_5 on the set of its Sylow 5-subgroups gives a homomorphism $\phi : S_5 \to S_X$ with trivial kernel.

(c) (10 points) Describe the action of a transposition in S_X acting on the cosets $S_X/\mathrm{im}(\phi)$. (Recall that a transposition is a two-cycle.)

Qualifying Exam. ODE

This is a closed book, closed notes exam. To receive full credit it is sufficient to provide correct solutions to any four out of the proposed five problems.

QE ID:						
Score:	1	2	3	4	5	$\sum$

Problem 1 (20 pts)**.** Write down the differential equation for the function $y(x)$ such that the family of its solutions parametrize the family of all circles tangent to the line $y = 0$. Justify your answer.

Problem 2 (20 pts)**.** Under which conditions on $f(x)$ all the solutions to the equation

$$y'' + y = f(x)$$

remain bounded for $x \to +\infty$? Justify your answer.

Problem 3 (20 pts)**.** Solve the system of differential equations:

$$\begin{cases} \dot{x} = 4x - y \\ \dot{y} = 3x + y - z \\ \dot{z} = x + z \end{cases}$$

Problem 4 (20 pts)**.** Let $y_1(x) = x^2$, $y_2(x) = 1 - x$ and $y_3 = 1 - 3x$ be three solutions of some second order inhomogeneous linear differential equation. Find the solution of this equation, satisfying the initial conditions $y(0) = 2$, $y'(0) = 0$.

Problem 5 (20 pts)**.** For which values of $a \in \mathbb{R}$ the equation

$$y'' + a^2 y = \sin 4x \cos 2x$$

has only one π-periodic solution?

Topology Qual Exam

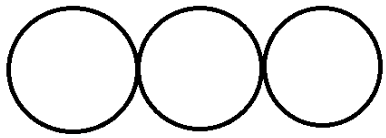
August 12, 2022

ID #:

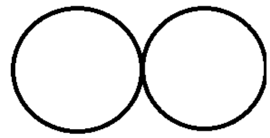
1) (15 pts) Show that a torus T^2 is not homotopy equivalent to a Klein bottle K^2.

2) (15 pts) Construct a space $\mathcal{X}$ with $H_2(\mathcal{X}) = \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_1(\mathcal{X}) = \mathbb{Z} * \mathbb{Z}_2$. Verify your answer.

3) (15 pts) Let X and Y be as follows.



X



Y

3a) Is there a covering map $\Pi : X \to Y$? If yes, describe it on the figure X. If not, show that X cannot be a covering space for Y.

3b) If there exists such covering, what is the corresponding subgroup to this covering space in $\pi_1(Y)$?

(Find $\Pi_*(\pi_1(X))$ in $\pi_1(Y)$).

4) (15 pts) Let $\mathbf{T}^2 = S^1 \times S^1$ be the 2-torus. Let $\mathcal{X}$ be the space obtained by collapsing one of the meridian circles $S^1 \times \{p\}$ in $\mathbf{T}^2$.

4a) Write a CW-decomposition for $\mathcal{X}$.

4b) Compute $\pi_1(\mathcal{X})$.

5) (20 pts) Give an example of two spaces with identical homology groups but they are not homotopy equivalent. Verify your answer.

6) (20 pts) Let $\mathcal{X} = \mathbf{S}^1 \vee \mathbf{S}^1$. Compute $H_i(\mathbf{S}^3 - \mathcal{X})$ for $i = 0, 1, 2, 3$.

(Hint: Use wedge point as the point at infinity)

Combinatorics and Graph Theory QE

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all five problems. Show your work and justify all statements.

Problem 1 (20 points).

(a) (10 points) Count the number of anagrams of the following string (do not simplify your answer):

MULTINOMIALSCOUNTANAGRAMS

(b) (10 points) Use the inclusion-exclusion principle to count the number of *surjective functions* from the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ to the set $Y = \{y_1, y_2, y_3\}$.

Problem 2 (30 points). Recall that the Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) (10 points) Show that F_{n+1} is the number of k-tuples $(\alpha_1, \ldots, \alpha_k)$ with $k \leq n$ such that each $\alpha_i \in \{1, 2\}$ and $\sum_{i=1}^{k} \alpha_i = n$.

(b) (10 points) Show that F_{n+2} is the number of subsets of $\{1, \ldots, n\}$ that do not contain two consecutive integers.

(c) (10 points) Consider the rational function $\varphi(z) := \dfrac{z}{1 - z - z^2}$, and let $\varphi^{(n)}(z)$ denote the n-th derivative of $\varphi(z)$. Show that $\varphi^{(n)}(0) = n! \cdot F_n$ for each $n \geq 0$.

Problem 3 (20 points). Let F be a forest with n vertices and k connected components such that $n > k > 0$. Let $V(F)$ denote the set of vertices of F.

(a) (10 points) Compute $\sum_{v \in V(F)} \deg(v)$ in terms of n and k.

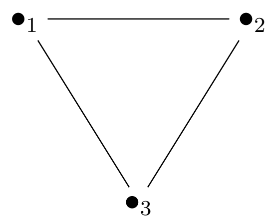
(b) (5 points) Show that the average degree of a vertex in F is strictly smaller than 2.

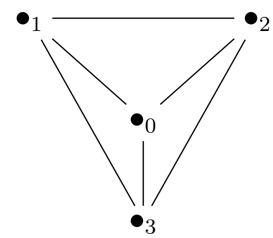
(c) (5 points) Conclude that every forest has at least one leaf.

Recall that a <u>tree</u> is a finite undirected connected graph with no cycles, a <u>forest</u> is a finite union of trees, and a <u>leaf</u> is a vertex of degree 1.

Problem 4 (30 points). Let $n \geq 3$, and let $\hat{C}_n$ be the "pyramid graph" with $n+1$ vertices obtained from the cycle graph C_n by adding one new vertex labelled 0 that is connected to each old vertex of C_n by one edge. For example, for $n = 3$ we have:

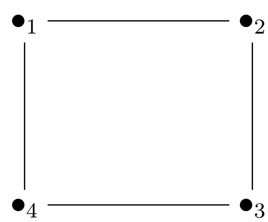
C_3 $\hat{C}_3$

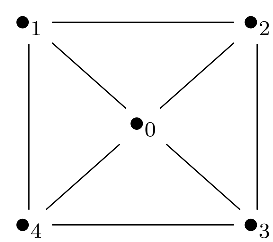




And for $n = 4$ we have:

C_4 $\hat{C}_4$





(a) (5 points) Show that the chromatic number of C_n is 2 for n even, and 3 for n odd.

(b) (10 points) Find the chromatic number of $\hat{C}_n$.

(c) (5 points) Show that the that the chromatic polynomial of C_n is $\chi_n(z) = (z-1)^n + (-1)^n(z-1)$.

(d) (10 points) Find the chromatic polynomial of $\hat{C}_n$.

Recall that the chromatic number of a graph G is the smallest number k such that the vertices of G can be colored properly with k colors, i.e., with no two adjacent vertices sharing the same color. The chromatic polynomial is a single-variable polynomial $\chi(z)$ so that $\chi(k)$ counts the number of proper colorings of G with k colors for each k.

MATH 6319 - Principles and Techniques of Applied Mathematics
Ph. D Qualifying Examinations
Summer 2022
Felipe Pereira & V. Ramakrishna

REQUIRED: Please enter the unique ID created for you for the QEs *on this line:*

It is recommended that you enter the ID, from the previous line, on each page of your answers.

CAUTION: Do **NOT** enter your name or UTD ID anywhere.

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** Compute in closed form the eigenvalues of the matrix $A = (i+j)$. Explain your work (i.e., state all results you will use precisely and at the point of usage).

- **Q2** State the Crabtree-Haynsworth formula. Derive it by using a result on the Schur complement in a threefold product of matrices.

- **Q3** State the law of complementary nullities. Use it to find the structure of the following submatrices $\begin{pmatrix} s_{41} & s_{42} & s_{43} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{66} \end{pmatrix}$ and $\begin{pmatrix} s_{13} & s_{14} & s_{15} & s_{16} \\ s_{23} & s_{24} & s_{25} & s_{26} \\ s_{33} & s_{34} & s_{35} & s_{36} \\ s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix}$ where S is the inverse of a 6×6 tridiagonal matrix T. You may assume that none of the entries of T on the diagonal and the first superdiagonal and subdiagonal is zero.

- **Q4** Let V be a complex pre-inner product space. Find a suitable quotient space V/W and suitable inner-product on it which is related to that on V. Justify your answer fully (i.e., identify W, explain why it is a subspace; explain why your inner product is well defined and why it is indeed an inner product etc.,) You may assume the contents of the previous result, but you must state them at the precise point of usage.

Q5 State and prove Fischer's inequality for positive semidefinite matrices.

Q6 Calcuate the convolution, in closed form, $f * f$, where f is characteristic function of the interval $[0, 1]$.

MATH 6322
Mathematical Foundations of Data Science
Ph. D Qualifying Examinations
Spring 2022
Yan Cao & V. Ramakrishna

REQUIRED: Please enter the unique ID created for you for the QEs *on this line:*

It is recommended that you enter the ID, from the previous line, on each page of your answers.

CAUTION: Do **NOT** enter your name or UTD ID anywhere.

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** Consider data $\{x_i, i = 1, \ldots, N\} \subseteq R^m$ which is to be clustered into K clusters, $\Gamma_a, a = 1, \ldots, K$. Define vectors $e_a \in R^n, a = 1, \ldots, K$ via $(e_a)_l = 1$ if x_l is in the ath cluster and 0 otherwise. Here $(e_a)_l, l = 1, \ldots, N$ is the lth component of the vector e_a. Next define a square $N \times N$ matrix $Z = \Sigma_{a=1}^K \frac{1}{|\Gamma_a|} e_a e_a^T$.

  - i) Explain why Z is positive semidefinite.
  - ii) Explain why Z is entrywise non-negative.
  - iii) Show $\mathrm{T}r(Z) = K$.
  - iv) Show $Ze = e$, where e is the vector of all ones in R^N.
  - v) Define a square $N \times N$ matrix D, concocted out the data set, such that the K-means cost function reduces to $\frac{1}{2}\mathrm{T}r(Z^T D)$.
  - vi) State what the Peng-Wei SDP relaxation of K-means is.

  $(4 + 4 + 4 + 4 + 4 + 5 = 25$ points)

- **Q2**

- i) Explain why the product of two positive definite kernels is also one.
- ii) What is one feature map which yields the inner product on a Hilbert Space as a kernel?
- iii) Define the Gaussian kernel.
- iv) Prove that the Gaussian kernel is indeed a kernel. All auxilliary results must be stated precisely at the point of usage.

$(7 + 7 + 5 + 6 = 25$ points)

- **Q3** State and prove a characterization, in terms of subgradients, of when a convex function is Lipschitz.

(25 points)

- **Q4**

  - i) Define a strongly convex function. Show that the function $f : \mathbf{R^n} \to \mathbf{R}, \mathbf{f}(\mathbf{x}) = \lambda \mid\mid \mathbf{x} \mid\mid^{\mathbf{2}}$ is strongly convex.
  - ii) Show that the global minima of strongly convex functions are unique.

(12+ 13= 25 points)

Q5 Show that $r_s(X) \in \{\mathrm{r}k(X), \mathrm{r}k(X) + 1\}$ (all auxilliary results must also be proved) The notation $r_s(X)$ stands for the semi non-negative rank of a matrix X.

Q6 In this question you will prove that a finite hypothesis class is agnostic PAC learnable. Do it via the following steps:

- Show that the true risk, $L_P(h)$, is the expected value of the empirical risk $L_S(h)$.

- If a training set is $\epsilon/2$ representative then for any output h_S of the ERM algorithm we have $L_P(h_S) \leq min_{\tilde{h} \in H} L_P(\tilde{h}) + \epsilon$.
- Show, using appropriate large deviation inequalities, that if H is a finite hypothesis class and l is a loss-function with values in $[0, 1]$, then H has the uniform convergence property and therefore that it is agnostic PAC learnable.

$(7 + 7 + 11 = 25$ points$)$