Qualifying Exam
Math 6301 January 2022
Real Analysis

QE ID______________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. (i) Is any compact subset of $\mathbb{R}$ Lebesgue measurable? Justify your answer.

   (ii) Is any compact subset of $\mathbb{R}$ Jordan measurable? Justify your answer.

2. Does there exist a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(y)$ is measurable for any $y \in \mathbb{R}$? Justify your answer.

3. Does the convergence in $L^{\frac{5}{2}}([0,1])$ imply the convergence in measure? Justify your answer.

4. (i) Show that

$$\int_{[0,1]} \Big(\int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \Big) dx = \frac{\pi}{4} \text{ and } \int_{[0,1]} \Big(\int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \Big) dy = -\frac{\pi}{4}.$$

   (ii) Does this contradict Fubini's theorem? Justify your answer.

Good luck!

Jniversity of Texas at Dallas

Functional Analysis I

Qualifying Exam, January 2022

QE ID:_______________

Problem 1.

(a) Formulate Kakutani's Theorem.

(b) Show that if $\mathbb{E}$ is a reflexive Banach space and $\mathbb{F}$ is a Banach space then for every bounded linear operator $L : \mathbb{E} \to \mathbb{F}$ the set

$$\{T(x) : x \in \mathbb{E},\ \|x\| \leq 1\}$$

is closed in $\mathbb{F}$.

Problem 2. Let $\mathbb{E}$ be the Euclidean space $\mathbb{R}^n$ and $\varphi : \mathbb{E} \to (-\infty, \infty]$ be given by

$$\varphi(x) := \begin{cases} \frac{1}{1-\|x\|} & \text{if } \|x\| < 1, \\ \infty & \text{if } \|x\| \geq 1. \end{cases}$$

(a) Check if φ is a convex function.

(b) Compute $\varphi^* : \mathbb{E}^* \to (-\infty, \infty]$.

Problem 3. Let $\mathbb{E}$ be a Banach space and $T : \mathbb{E} \to \mathbb{E}^*$ be a linear operator such that

$$\forall_{x \in \mathbb{E}} \quad \langle Tx, x \rangle \geq 0. \tag{1}$$

Use closed graph theorem to show that T is a bounded linear operator.

Problem 4. Let $\mathbb{E}$ be an infinite-dimensional Banach space and $S := \{x \in \mathbb{E} : \|x\| = 1\}$. Find the closure of the set S with respect to the weak topology $\sigma(\mathbb{E}, \mathbb{E}^*)$ in $\mathbb{E}$.

Complex Analysis Qualifying Exam

Spring 2022

Friday, January 7, 2022

1. [**25 points**] **True** or **false** (Justification is needed):

(a) If $f(z)=\sum f_n z^n$ and $g(z)=\sum g_n z^n$ define holomorphic functions on a neighborhood of the closed unit disk $D = \{z : |z| \le 1\}$ then $h(z)=\sum f_n g_n z^n$ also defines a holomorphic function on a neighborhood of D.

(b) The function $f(z)=\dfrac{\sin(8z)-6z}{\cos z - 1 + z^2/2}$ has a pole of order one at $z=0$.

(c) If z_1, z_2, z_3 are the vertices of an equilateral triangle then $z_1^2+z_2^2+z_3^2 = z_1z_2+z_3z_1+z_2z_3$.

2. [**25 points**] For each of the following, explain why there cannot be an entire function $f : C \to C$ with the stated properties.

(a) $\int_\gamma f(z)\,dz = 5$, where γ is the positively oriented circle $|z| = 1$.

(b) $f(yi) = yi$ for $0 \le y \le 1$ and $f(7+2i) = 2i$.

(c) $|f(x+yi)| = e^{-(x^4+y^4)}$ for all $x+yi \in C$.

(d) f has a zero of order 5 at the origin and $\int_\gamma f\left(\frac{1}{z}\right)dz = 2\pi i$, where γ is the positively oriented circle $|z| = 1$.

3. [**25 points**] Suppose that $f(z)$ is holomorphic on $D=\{z|\,|z|<1\}$ and that $|f(z)|<1|$. Show that

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}\cdot f(z)}\right| \le \left|\frac{z-z_0}{1-\bar{z}_0\cdot z}\right|.$$

4. [**25 points**] Use the calculus of residues to evaluate the integral for each real number a

$$\int_0^\infty \frac{\cos(ax)}{x^4+5x^2+4}\,dx.$$

Verify all steps of the calculation.

Algebra Qualifying Exam

Spring 2022

QE ID:

Instructions. Write your QE ID above and solve all four problems. Show your work and justify all statements.

Problem 1 (25 points).

(a) Let G be a group, and suppose $g \in G$ has order ab with $\gcd(a, b) = 1$. Prove that $g = xy$ for some $x, y \in G$ with orders a and b, respectively.

(b) Let G be a group and let $x, y \in G$. Show that xyy, yxy, and yyx all have the same order.

Problem 2 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements x, y, z subject to the relations

$$\begin{aligned} 12x + 2y - 12z &= 0 \\ 69x + 9y - 87z &= 0 \\ -84x - 12y + 96z &= 0. \end{aligned}$$

Problem 3 (25 points). Prove that there $\frac{1}{n}\sum_{d|n}\phi(d)k^{n/d}$ different to build a circular necklace using n beads that can each be one of k different colors, up to circular symmetry. Here, $\phi(d)$ is Euler's totient function, defined by $\phi(d) = |\{i : 1 \le i \le d, \gcd(d,i) = 1\}|$.

Problem 4 (25 points). Let p be prime, and let $G = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$.

(a) Compute the order of G.

(b) Find a Sylow p-subgroup of G.

Qualifying Exam. ODE

This is a closed book, closed notes exam. To receive full credit it is sufficient to provide correct solutions to any four out of the proposed five problems.

<table>
<tr><td>QE ID:</td><td colspan="6"></td></tr>
<tr><td rowspan="2">Score:</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>$\sum$</td></tr>
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Problem 1 (20 pts)**.** For which integer values of n the equation

$$x^{(n)} = f(t, x, \dot{x}, \cdots, x^{(n-1)})$$

may have $x_1(t) = t$ and $x_2(t) = \sin t$ as its solutions?

Problem 2 (20 pts)**.** For which values of a and b every solution of the equation

$$y'' + ay' + by = 0$$

have infinitely many zeros?

Problem 3 (20 pts)**.** Solve the system of differential equations:

$$\begin{cases} \dot{x} = z - x - y \\ \dot{y} = x - y - z \\ \dot{z} = -y \end{cases}$$

Problem 4 (20 pts)**.** Find the general solution of the equation

$$\ddot{x} - 2\dot{x} + x = e^t + \sin t$$

Problem 5 (20 pts)**.** For which values of $a \in \mathbb{R}$ at least one solution of the equation

$$y''' + a^2 y' = \cos(at)\cos(2t)$$

is bounded for $t \geqslant 0$?

MATH 6309 (DIFFERENTIAL GEOMETRY)
JANUARY 2022 QUALIFYING EXAM

QE ID:

There are 4 problems. Each problem is worth 25 points. The total score is 100 points.

Show all your work to get full credits.

Problem 1. Let $M_2(\mathbb{R})$ denote the space of 2×2 matrices with real entries. Let N be the subset of $M_2(\mathbb{R})$ consisting of all non-zero matrices A such that $\det(A) = 0$. Show that N is a 3-dimensional manifold.

Problem 2. Let N and M be smooth connected manifolds and let $f : N \to M$ be a non-constant smooth map. Show that there exists $p \in N$ such that $f_* : T_pN \to T_{f(p)}M$ is not equal to the zero map. [Hint: apply the formula $f_* \frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^m \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}\Big|_q$.]

Problem 3. Let $\omega^1, \dots, \omega^k$ be covectors on a finite-dimensional vector space V.

(a) Show that $\omega^1, \dots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

(b) Suppose $\omega^1, \dots, \omega^k$ are linearly independent, and so is the collection of covectors $\eta^1, \dots, \eta^k \in V^*$. Prove that $\operatorname{span}(\omega^1, \dots, \omega^k) = \operatorname{span}(\eta^1, \dots, \eta^k)$ if and only if there exists $c \in \mathbb{R} \setminus \{0\}$ such that $\omega^1 \wedge \cdots \wedge \omega^k = c\, \eta^1 \wedge \cdots \wedge \eta^k$.

Problem 4. Let $\omega = ydx - xdy$ be a 1-form on $\mathbb{R}^2$, and let $\sigma = \frac{1}{x^2+y^2}\,\omega$ be a 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$.

(a) Is ω closed? Is ω exact?

(b) Is σ closed? Is σ exact?

MATH 6310 (TOPOLOGY) – JANUARY 2022 QUALIFYING EXAM

QE ID:

There are 4 problems. Each problem is worth 25 points. The total score is 100 points.

Show all your work to get full credits.

Problem 1. (a) Let $f : X \to Y$ be a continuous bijection between topological spaces, and suppose that X is compact and Y is Hausdorff. Show that f has a continuous inverse.

(b) Let $p : \tilde{X} \to X$ be a covering map and $A \subset X$, and let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p\,|_{\tilde{A}} \colon \tilde{A} \to A$ is a covering map.

Problem 2. (a) Give the definition of a retraction of a topological space onto a subspace.

(b) Show that if $r : X \to A$ is a retraction and $x_0 \in A$ then

(i) the induced homomorphism $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is a surjective map,

(ii) $\pi_1(X, x_0)$ has a subgroup isomorphic to $\pi_1(A, x_0)$.

(c) Show that there is no retraction from the solid torus $S^1 \times D^2$ to its boundary torus $S^1 \times S^1$.

Problem 3. (a) Give the definition of a cell complex.

(b) Let X be the quotient space of the unit 2-sphere S^2 formed by identifying the north and south poles.

(i) Find a cell complex structure of X.

(ii) Compute the fundamental group of X.

Problem 4. (a) Give the definition of a standard k-simplex.

(b) Let Δ^k be the standard k-simplex, and X^k be the Δ-complex $\Delta^k/\sim$ where we identify all faces of the same dimension. Compute the homology groups of X^k.

MATH 6319 - Ph. D Qualifying Examinations
Spring 2022
Felipe Pereira & V. Ramakrishna

REQUIRED: Please enter the unique ID created for you for the QEs *on this line:*

It is recommended that you enter the ID, from the previous line, on each page of your answers.

CAUTION: Do **NOT** enter your name or UTD ID anywhere.

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** Compute in closed form the eigenvalues of the matrix $A = (i - j)$. Explain your work (i.e., state all results you will use precisely and at the point of usage).

- **Q2** State the Crabtree-Haynsworth formula. Derive it by using a result on the Schur complement in a threefold product of matrices.

- **Q3**

Show that on a pre-inner product space that the Gram matrix of a finite set of vectors is psd. Use it to prove that the CBS inequality holds in pre-IPS as well. If the space is actually an inner product space, state and prove a condition characeterizing when the Gram matrix is is positive definite. Use it to characeterize equality in the CBS inequality.

- **Q4** Let V be a complex pre-inner product space. Find a suitable quotient space V/W and suitable inner-product on it which is related to that on V. Justify your answer fully (i.e., identify W, explain why it is a subspace; explain why your inner product is well defined and why it is indeed an inner product etc.,) You may assume the contents of the previous result, but you must state them at the precise point of usage.

Q5 Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier tarnsform.

Q6 Calcuate the convolution, in closed form, $f * f$, where f is characteristic function of the interval $[0, 1]$.