

**Qualifying Exam (closed book)**  
**Math 6301 August 2021**  
**Real Analysis**

QE ID \_\_\_\_\_

**Instructions:** *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. (a) Let  $A \subset [0, 1] \times [0, 1]$  be a non-measurable set (with respect to the planar Lebesgue measure), and let  $\mu_3(\cdot)$  be the Lebesgue measure on  $\mathbb{R}^3$ . Put

$$E := \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1\}. \quad (1)$$

Is the subset

$$B := \{(x, y, 0) \in E : (x, y) \in A\}$$

measurable with respect to  $\mu_3(\cdot)$ ? Justify your answer.

- (b) Does there exist an open subset  $K \subset \mathbb{R}^3$  such that  $\mu_3(K) = 0$ ? Justify your answer.

2. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function admitting the third derivative at any  $x \in [0, 1]$ .

(i) Is  $f'''$  a continuous function? Justify your answer.

(ii) Is  $f'''$  a Lebesgue measurable function? Justify your answer.

3. Assume that  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing function and let  $g'$  be its derivative (existing almost everywhere). If  $g$  is not differentiable at some  $x \in [\alpha, \beta]$ , put  $g'(x) = 2021$ .

(a) Show that  $g'$  is integrable on  $[\alpha, \beta]$ ;

(b) Does  $g'$  satisfy the formula

$$\int_{[\alpha, \beta]} g' d\mu = g(\beta) - g(\alpha) \quad (2)$$

(here  $\mu$  stands for the line Lebesgue measure)? Justify your answer.

4. Is the set of irrational numbers belonging to the segment  $[0, 1]$  Jordan measurable? Justify your answer.

**Good luck!**



University of Texas at Dallas

# QUALIFYING EXAM

Functional Analysis	
<b>Exam Date:</b> August 9, 2021	<b>Identification Number:</b> .....

## Instructions:

1. Use the space provided to write your solutions in this booklet to write the final (neat, elegant and precise) version of your solutions.
3. Provide all the necessary definitions and state all the theorems that you need for your solutions.

Question	Weight	Your Score	Comments
1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

**Problem 1.** Consider the space  $\mathbb{E} := C([a, b]; \mathbb{R})$  of continuous functions on the interval  $[a, b]$  equipped with the norm

$$\|\varphi\| := \left( \int_a^b |\varphi(t)|^2 dt \right)^{\frac{1}{2}} + \sup_{t \in [a, b]} |\varphi(t)|, \quad \varphi \in \mathbb{E}.$$

- (a) Show that  $\mathbb{E}$  is a Banach space;
- (b) Consider the linear functional  $f : \mathbb{E} \rightarrow \mathbb{R}$  given by

$$f(\varphi) := \varphi\left(\frac{a+b}{2}\right), \quad \varphi \in \mathbb{E}.$$

Compute  $\|f\|$ .

**SOLUTION:**

**Problem 2.** Let  $\mathbb{E}$  be a Banach space and  $A : D(A) \subset \mathbb{E} \rightarrow \mathbb{E}$  a densely-defined unbounded operator.

- (a) Show that  $A$  is a closed operator if and only if it is weakly closed (i.e. the graph of  $A$   $\text{Gr}(A) := \{(x, Ax) \in \mathbb{E} \times \mathbb{E} : x \in D(A)\}$  is weakly-closed).
- (b) Put  $\mathbb{E} := C([a, b]; \mathbb{R})$  (equipped with the usual sup-norm  $\|\varphi\|_\infty := \sup_{t \in [a, b]} |\varphi(t)|$ ) and define  $A : D(A) \subset \mathbb{E} \rightarrow \mathbb{E}$  by  $(A\varphi)(t) := \varphi'(t)$ ,  $\varphi \in \mathbb{E}$ , where  $D(A)$  stands for the subspace of all continuously differentiable functions in  $\mathbb{E}$ . Check if the operator  $A$  is closed.

**SOLUTION:**

**Problem 3.** Let  $\mathbb{E}$  be Banach space and  $K \subset \mathbb{E}$  be a compact set.

(a) Show that  $\overline{\text{conv}(K)}$  is compact.

(b) Show that if  $K$  has a nonempty interior then  $\mathbb{E}$  is finite-dimensional.

**Solution:**

**Problem 4:** Let  $\mathbb{E} := L^1([a, b]; \mathbb{R})$  and  $\mathbb{F} := L^2([a, b]; \mathbb{R})$  be Banach spaces and  $A : \mathbb{E} \rightarrow \mathbb{F}$  be a linear operator given by

$$(A\varphi)(x) := \int_a^x \varphi(t) dt, \quad \varphi \in E, \quad x \in [a, b].$$

Compute the norm  $\|A\|$ .

**SOLUTION:**

# Complex Analysis Qualifying Exam

Summer 2021

Friday, August 13, 2021

1. [20 points] **TRUE** or **FALSE**. (Justification is needed):

- (a) Let  $\Omega$  be a region containing  $\bar{D} = \{z \mid |z| \leq 1\}$ . Let  $f : \Omega \rightarrow \mathbb{C}$  be analytic and  $M > 0$  be a constant such that  $|f(z)| \geq M$  for all  $z$ ,  $|z| = 1$  and  $|f(0)| < M$ . Then  $f(z)$  has at least one zero in  $D = \{z \mid |z| < 1\}$ .
- (b)  $\int_{\gamma} \operatorname{Re} z dz$  is independent of the path  $\gamma$  between  $z_0 = 0$  and  $z_1 = 1 + i$ .
- (c) If  $f : D = \{z \mid |z| < 1\} \rightarrow \mathbb{C}$  is analytic, then  $|f'(0)| \leq 1$ .
- (d) If  $\{f_n\}_{n=0}^{\infty}$  is a sequence of analytic functions on a set  $D$  that uniformly converges to  $f$ , then  $f$  is analytic on  $D$ .

2. [20 points] Which of the three functions

$$y \cos x, \quad x \sin y - y \sin x, \quad x^4 - 6x^2 y^2 + y^4$$

can be the real part of some analytic function? For all valid candidates find the imaginary part of the corresponding analytic function.

3. [20 points] Let  $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$  be a polynomial with real coefficients.

Let  $Q(z) = z^m + b_1 z^{m-1} + \cdots + b_m$ ,  $m = n(n-1)/2$  be the polynomial whose zeros are sums of pairs of zeros of  $P$ . Prove that all zeros of  $P$  have negative real parts if and only if both polynomials  $P$  and  $Q$  have all the coefficients positive.

4. [20 points] Find all entire functions  $f$  such that  $|f(z)| \leq 1/|z|$  for all  $z$ ,  $|z| > 1$ .

5. [20 points] Compute the integral  $\int_0^{2\pi} \frac{\cos^2 x}{5 - 4 \cos x} dx$  using complex methods.



QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. There is one bonus problem. Show your work and justify all statements.

**Problem 1 (25 points).**

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements  $x, y, z$  subject to the relations

$$8x - 10y + 4z = 0$$

$$72x - 56y + 24z = 0$$

$$-40x + 40y - 16z = 0.$$

**Problem 2 (25 points).**

Let  $S_4$  be the symmetric group on  $\{1, 2, 3, 4\}$ . Let  $S_4$  act on the elements

$$\{\{1, 2\}, \{3, 4\}\}, \quad \{\{1, 3\}, \{2, 4\}\}, \quad \{\{1, 4\}, \{2, 3\}\}.$$

by  $\sigma(\{\{i, j\}, \{k, \ell\}\}) = \{\{\sigma(i), \sigma(j)\}, \{\sigma(k), \sigma(\ell)\}\}$  for  $\sigma \in S_4$ .

(a) (10 points) List the elements in  $\ker(\phi)$ .

(b) (15 points) Show that the group  $S_4/\ker(\phi)$  is isomorphic to  $S_3$ .

**Problem 3 (25 points).**

- (a) (15 points) Find all 2-Sylow subgroups of the alternating group  $A_4$  (recall that the alternating group  $A_4$  is the unique subgroup of the symmetric  $S_4$  of index 2).

- (b) (10 points) Find a finite group  $G$  and a divisor  $d$  of its order  $|G|$  such that  $G$  has no subgroup of order  $d$ .

**Problem 4 (25 points).** Let  $F$  be the set of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ -3b & a \end{bmatrix}$  with  $a, b \in \mathbb{R}$  (where  $\mathbb{R}$  is the set of real numbers). Show that  $F$  is a field under the usual matrix operations.

**Bonus (10 points).** Recall that the symmetric group  $S_4$  has the 4-dimensional complex representation  $V$ , where a permutation acts by its corresponding  $4 \times 4$  permutation matrix. The subgroup

$$\{e, (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$$

of  $S_4$  is isomorphic to the dihedral group with 8 elements  $D_8$ , and so  $V$  is a representation of  $D_8$ . Decompose  $V$  as a direct sum of irreducible representations for  $D_8$ .

Qualifying Exam: Ordinary Differential Equations I, August 2021

QE ID: \_\_\_\_\_

**THIS IS A CLOSED BOOK, CLOSED NOTES & NO CALCULATOR EXAM**

Problems count 25 points each

Give clear and complete answers with full details in proofs

1. Consider the initial value problem  $x' = x^{1/5}$  and  $x(0) = x_0$  for  $t \geq 0$ .

a) Is the solution to the above IVP unique if  $x_0 = 1$ ?

b) Is the solution to the above IVP unique if  $x_0 = 0$ ?

Clearly justify/prove all your answers.

2. Show that any solution of

$$x'' + x + x^3 = 0$$

exists for all  $t \in R$ . Can the same be said about the equation

$$x'' + x' + x + x^3 = 0 ?$$

Clearly justify/prove your answers.

3. Prove that if  $\Phi(t)$  is a fundamental set of solutions of  $x' = A(t)x$ , where  $A(t)$  is a periodic matrix with  $A(t) = A(t + T)$ , then  $\Phi(t + T)$  is also a fundamental set of solutions of  $x' = A(t)x$ . Furthermore, prove that there exists a non-singular periodic matrix  $P(t)$ , with period  $T$  and a constant matrix  $R$  such that  $\Phi(t) = P(t)e^{tR}$ . Clearly justify your answers.

4. Consider the BVP  $Lx = \lambda x$ , where  $t \in [0, 1]$ ,  $x(0) + x(1) = 0$ , and  $Lx = i \frac{d}{dt}x + t^2x$ .

a.) Is the defined operator  $L$  self-adjoint?

b.) Find the eigenvalues of above BVP.

c.) Find the eigenfunctions of above BVP. Are the found eigenfunctions orthogonal?

Clearly justify/prove all your answers.

**MATH 6310 (TOPOLOGY) – AUGUST 2021 QUALIFYING EXAM**

QE ID:

There are 4 problems. Each problem is worth 25 points. The total score is 100 points.

Show all your work to get full credits.

**Problem 1.** The complex projective  $n$ -space  $\mathbb{C}P^n$  is the space of complex lines through the origin in  $\mathbb{C}^{n+1}$ .

- a) Show that  $\mathbb{C}P^n$  is obtained from  $\mathbb{C}P^{n-1}$  by attaching a  $2n$ -dimensional cell.
- b) Find a cell structure of  $\mathbb{C}P^n$ .
- c) Compute the homology groups of  $\mathbb{C}P^n$ .



- Problem 2.** a) Construct an explicit deformation retraction of  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  onto  $\mathbb{S}^{n-1}$ .
- b) Compute the fundamental group of  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ , where  $n \geq 1$  is an integer.
- c) Use the fundamental group to show that  $\mathbb{R}^n$  is not homeomorphic to  $\mathbb{R}^2$  if  $n \neq 2$ .

**Problem 3.** The Klein bottle is obtained as a quotient space of  $[0, 1] \times [0, 1]$  by the equivalence relations  $(s, 0) \sim (s, 1)$  and  $(0, t) \sim (1, 1 - t)$ , for all  $s, t \in [0, 1]$ .

a) Compute the fundamental group of the Klein bottle by applying the van Kampen theorem.

b) Compute the homology groups of the Klein bottle.

**Problem 4.** Use homology theory to show that  $\mathbb{R}^m$  and  $\mathbb{R}^n$  are not homeomorphic if  $m \neq n$ .

**MATH 6309 (DIFFERENTIAL GEOMETRY)  
AUGUST 2021 QUALIFYING EXAM**

QE ID:

There are 4 problems. Each problem is worth 25 points. The total score is 100 points.  
Show all your work to get full credits.

**Problem 1.** Let  $M$  be the set of all straight lines in  $\mathbb{R}^2$ . Show that  $M$  is a smooth 2-dimensional manifold.

**Problem 2.** The real projective plane  $\mathbb{R}P^2$  is the space of all straight lines through the origin in  $\mathbb{R}^3$ . Show that the map

$$f : \mathbb{R}P^2 \rightarrow \mathbb{R}, \quad [(x : y : z)] \mapsto \frac{xy + yz + zx}{x^2 + y^2 + z^2}$$

is well-defined and smooth.

**Problem 3.** Show that the map

$$f : \mathbb{R}P^2 \rightarrow \mathbb{R}^4, \quad [(x : y : z)] \mapsto \frac{1}{x^2 + y^2 + z^2} (yz, zx, xy, y^2 - z^2),$$

is an immersion.

**Problem 4.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  and  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  be given by

$$f(x, y) = \left( \frac{4x}{x^2 + y^2 + 4}, \frac{4y}{x^2 + y^2 + 4}, \frac{x^2 + y^2 - 4}{x^2 + y^2 + 4} \right),$$
$$g(x, y, z) = x^2 + y^2 + z^2.$$

Compute  $f^*g_*$  and  $(g \circ f)_*$ , and compare the answers.





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# QUALIFYING EXAM

Nonlinear Analysis	
<b>Exam Date:</b> August 13, 2021	<b>Identification Number:</b> .....

## Instructions:

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Question	Weight	Your Score	Comments
1.	35		
2.	30		
3.	35		
Total:	100		

**Problem 1.** (a) Suppose that  $U \subset \mathbb{C}$  is an open bounded set,  $f : \bar{U} \rightarrow \mathbb{C}$  a continuous  $U$ -admissible map such that  $f|_U$  is analytic. Show that the Brouwer degree  $\deg(f, U)$  is exactly the number of zeros of  $f$  in  $U$ , counted with multiplicity.  
(b) Use the degree to prove the Fundamental Theorem of Algebra.

**SOLUTION:**

**Problem 2.** Assume that  $f : S^n \rightarrow S^n$ ,  $n \geq 1$ , is a continuous map homotopic to a constant map then:

- (a)  $f$  has a fixed-point, i.e. there is  $x_o \in S^n$  such that  $f(x_o) = x_o$ ;
- (b) there is a point  $x'_o \in S^n$  such that  $f(x'_o) = -x'_o$ .

**SOLUTION:**

**Problem 3.** (a) Prove the following Borsuk-Ulam theorem: *For every continuous map  $f : S^n \rightarrow \mathbb{R}^n$ , there exists  $x \in S^n$  such that  $f(x) = f(-x)$ .*  
(b) Use the degree to show that if  $m \neq n$  then the space  $\mathbb{R}^m$  is not homeomorphic to  $\mathbb{R}^n$ .

**Solution:**

INSTRUCTIONS. Write your QE ID above and solve all four problems. There is one bonus problem. Show your work and justify all statements.

## Problem 1 (25 points).

(a) (10 points) Let  $B(n, k)$  be the set of integer partitions whose Ferrers diagrams fit inside an  $n \times k$

box. For example, decorating the boxes in  $\lambda$  by a “\*”,  $\lambda =$

*	*	*	*	*
*	*	*	*	
*	*			
*	*			

is in  $B(5, 4)$ . Explain

why  $|B(n, k)| = \binom{n+k}{k}$ .

*Hint: interpret the partitions in  $B(n, k)$  as paths in  $\mathbb{Z}^2$  from  $(0, 0)$  to  $(n, k)$  using steps of the form  $(1, 0)$  and  $(0, 1)$ .*

For the remainder of this problem and in Problem 2, we will use the following polynomial analogues of the numbers  $n$ ,  $n!$ , and  $\binom{n}{k}$ :

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},$$


$$[n]!_q = [n]_q [n-1]!_q \text{ with initial condition } [0]!_q = 1, \text{ and}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

(b) (15 points) Use algebra to show that  $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$  is a *polynomial* in  $q$  by showing that it satisfies the recurrence

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q.$$

**Problem 2 (25 points).** Define the weight  $|\lambda|$  of a partition  $\lambda \in B(n, k)$  to be the number

of boxes it contains. For example,  $\lambda =$   in  $B(5, 4)$  has  $|\lambda| = 13$  (the boxes decorated by a “\*”). Give a combinatorial argument to show that the polynomial

$$\sum_{\lambda \in B(n, k)} q^{|\lambda|}$$

satisfies the same recurrence and initial conditions as  $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ .

**Problem 3 (25 points).**

(a) (10 points) Write a binomial coefficient for the number of integer points satisfying

$$x_1 + x_2 + x_3 = 40 \text{ if } x_1 \geq 5, x_2 \geq 3, x_3 \geq -4.$$

(b) (15 points) Use inclusion-exclusion to count the number of integer points satisfying

$$x_1 + x_2 + x_3 = 40 \text{ if } 0 \leq x_1, x_2, x_3 \leq 18.$$



**Problem 4 (25 points).**

For  $n \geq 3$ , let  $C_n$  be the cycle graph with  $n$  vertices. Show that the chromatic polynomial for  $C_n$  is  $(k-1)^n + (-1)^n(k-1)$ .

*(Recall that the chromatic polynomial of a graph  $G$  is the single-variable polynomial  $\chi_G$  with the property that  $\chi_G(k)$  counts the number of ways to properly color the vertices of  $G$  with  $k$  colors.)*

**Bonus (10 points).** Give a combinatorial proof of the identity

$$n(n+1)2^{n-2} = \sum_{i=1}^n i^2 \binom{n}{i}.$$

*(Non-combinatorial proofs will only receive partial credit.)*