Qualifying Exam
Math 6301 January 2021
Real Analysis

QE ID______________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Find the (planar) Lebesgue measure of the subset of the unit square in the plane consisting of the points both of whose Cartesian coordinates and both of whose polar coordinates are irrational. Justify your answer.

2. Let a function $h : [0,1] \to \mathbb{R}$ satisfy the following conditions:

   (i) h is differentiable in $(0,1)$;

   (ii) h admits one-sided derivatives at 0 and 1;

   (iii) there exists $M > 0$ such that $|h'(x)| \leq M$ for all $x \in [0,1]$.

   Show that:

   (a) h' is integrable;

   (b) $\int_{[a,b]} h' d\mu = h(b) - h(a)$.

3. Does there exist a non-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $|g|$ is a measurable function and $g^{-1}(\alpha)$ is a measurable set for each $\alpha \in \mathbb{R}$? Justify your answer.

4. Let $\{h_n : [0,1] \to \mathbb{R}\}$ (resp. $\{g_n : [0,1] \to \mathbb{R}\}$ be two sequence of measurable functions convergent in measure to h (resp. g). Assume that for each $n \in \mathbb{N}$, one has $h_n(x) = g_n(x)$ almost everywhere. Show that $h(x) = g(x)$ almost everywhere.

Good luck!

University of Texas at Dallas

QUALIFTYING EXAM

Functional Analysis	
Exam Date:	**Identification Number:**
January 4, 2021	....

Instructions:

1. Use the space provided to write your solutions in this booklet to write the final (neat, elegant and precise) version of your solutions.
3. Provide all the necessary definitions and state all the theorems that you need for your solutions.

Question	**Weight**	**Your Score**	**Comments**
1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

Problem 1. Let $\mathbb{E}$ be an infinite-dimensional Banach space. Show that:

(a) every non-empty set $\mathscr{U}$ in $\mathbb{E}$ that is weakly open with respect to $\sigma(\mathbb{E}, \mathbb{E}^*)$-topology, is unbounded;

(b) the space $\mathbb{E}$ equipped with $\sigma(\mathbb{E}, \mathbb{E}^*)$-topology is not metrizable.

SOLUTION:

Problem 2. Let $\mathbb{E}$ be a Banach space. Fix an element $x_o \in \mathbb{E}$ and for $r > 0$ we put

$$S_r := \{x \in \mathbb{E} : \|x - x_o\| = r\}.$$

Show that if the set S_r is compact then $\mathbb{E}$ is finite-dimensional.

SOLUTION:

Problem 3. Assume that $a \in \mathbb{R}^n$ is a fixed vector and $b \in \mathbb{R}$. Define the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi(x) = a \bullet x + b, \quad x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n,$$

where $x \bullet y := \sum_{k=1}^{n} x_k y_k$. Compute $\varphi^* : (\mathbb{R}^n)^* \to (-\infty, \infty]$. What is φ^{**}?

Solution:

Problem 4: Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces and $A : \mathbb{E} \to \mathbb{F}$ be a linear operator. Show that

$$A \in L(\mathbb{E}, \mathbb{F}) \quad \Leftrightarrow \quad \forall_{f \in \mathbb{F}^*} \;\; f \circ A \in \mathbb{E}^*.$$

SOLUTION:

Complex Analysis Qualifying Exam

Spring 2021

Friday, January 8, 2021

1. [**25 points**] **True** or **false** (Justification is needed):

(a) Let A_1 and A_2 be two points on the Riemann sphere which map to z_1 and z_2 respectively in the stereographic projection. The points A_1 and A_2 are antipodal if and only if $z_1 \cdot \overline{z_2} = -1$. ($z_1$, z_2 $\in \boldsymbol{C}$).

(b) The derivative of a fractional-linear transformation is never equal to zero.

(c) Suppose f and g are one to one analytic functions from unit disk $D = \{z \mid |z| < 1\}$ onto itself, such that $f(0) = g(0)$ and $f'(0) = g'(0)$. Then $f(z) = g(z)$ for all $z \in D$.

2. [**25 points**]
Given a function f in $\bar{C}$ with the only singularities at $z = 1$ as a simple pole and $z = -1$ as a second order pole, with the residuum equal to 1 and 0 respectively. Let in addition $f(0) = -1, f(2) = \frac{17}{9}$.

(a) Determine the function f.

(b) Expand the function f in the Laurent series in a punctured neighborhood of $z = \infty$.

(c) Expand the function f in the Laurent series centered at $z = 1$, such that the series converges at $z = 0$.

3. [**25 points**] Suppose $f : \boldsymbol{C} \to \boldsymbol{C}$ is entire with $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(a) Show that if $a_n \neq 0$ for infinitely many n then f has an essential singularity at infinity.

(b) Show that if f is injective, then $f(z) = a_0 + a_1 z$ with $a_1 \neq 0$.

4. [**25 points**] Use the calculus of residues to evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2 - x + 1)^2}.$$

Verify all steps of the calculation.

Algebra Qualifying Exam

SPRING 2021

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. There is one bonus problem. Show your work and justify all statements.

Problem 1 (25 points).

(a) (5 points) Let G be a group and X be a set. Define what it means for G to act on X.

(b) (10 points) Let X be a finite set with n elements and let π be a permutation acting on X such that $\pi \circ \pi$ is the identity, and such that π has m fixed points. Show that $n - m$ is even.

(c) (10 points) Let G be a finite group of even order. Use part (a) and the permutation $\pi : g \mapsto g^{-1}$ to show that there exists an element of order 2 in G.

Problem 2 (25 points). Let S_5 be the symmetric group of permutations of $\{1, 2, 3, 4, 5\}$.

(a) (10 points) Find the number of Sylow 3-subgroups of S_5, the number of Sylow 5-subgroups of S_5, and classify these subgroups up to isomorphism.

(b) (10 points) Show that there are 15 Sylow 2-subgroups of S_5.

(c) (5 points) Show that each Sylow 2-subgroup is isomorphic to the dihedral group of order 8.

Problem 3 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Let G and H be the finitely-generated abelian groups defined by

$$G = \left\langle a, b, c : \begin{array}{r} 10b - 4c = 0 \\ 12a + 6b + 6c = 0 \\ 60a + 12b + 36c = 0 \end{array} \right\rangle \text{ and } H = \left\langle x, y, z : \begin{array}{r} 2x + 2y \qquad\quad = 0 \\ 54x + 24y + 18z = 0 \\ 48x + 24y + 12z = 0 \end{array} \right\rangle .$$

Determine if G and H are isomorphic.

Problem 4 (25 points).

(a) (5 points) Define what it means for a subgroup H of a group G to be normal.

(b) (20 points) Let F be a free group and for a fixed integer n, let N be the subgroup generated by the set $\{x^n : x \in F\}$. Show that N is normal in F.

Bonus (10 points). Let D_8 be the dihedral group of order 8.

(a) (5 points) Draw the subgroup lattice of D_8 and circle its normal subgroups.

(b) (5 points) Find the group of inner automorphisms of D_8 and the group of all automorphims of D_8.

Qualifying Exam
Ordinary Differential Equations I, January 2021

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 25 points each

Give clear and complete answers with full details in proofs

QE ID: ________________

1. Show that the initial value problem $x' = x^{(1/7)}$ with $x(0) = 0$ has infinitely many different solutions.

2. For any finite initial condition, show that solution to $x'' + x + x^3 = 0$ exists for all $t \in R$
   Hint: You may want to use proof by contradiction.

3. Consider the differential operator $L = t^2 \frac{d^2}{dt^2} + 2t\frac{d}{dt}$, for $t \in [1, e]$
   Prove that the boundary value problem $Lu = \lambda u$ with $u(1) = u(e) = 0$ can have only real eigenvalues.

4. (a.) Define the successive approximate solutions ϕ_k of $x' = f(t, x)$ with $x(\tau) = \xi$.
   (b.) If $f(t, x)$ is a continuous function of t and Lipschitz continuous function of x, then give a complete proof that ϕ_k's converge uniformly to a unique solution ϕ of the above IVP.

MATH 6310 (TOPOLOGY) – JANUARY 2021 QUALIFYING EXAM

QE ID:

There are 5 problems. Each problem is worth 20 points. The total score is 100 points. Show all your work to get full credits.

Problem 1. Let $B^n = \{x \in \mathbb{R}^n : ||x|| < 1\}$ be the unit open ball in $\mathbb{R}^n$. Show that B^n is homeomorphic to $\mathbb{R}^n$.

Problem 2. Consider $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi ti}$. Show that p is a covering map, but $p\,|_{[-1,1]} \colon [-1, 1] \to S^1$ is not.

Problem 3. Suppose X is a path-connected, locally path-connected space such that $\pi_1(X) = \mathbb{Z}_{2021}$. Show that every continuous map $f : X \to S^1$ is nullhomotopic.

Problem 4. Use homology to show that every continuous map $f : D^{2021} \to D^{2021}$ has a fixed point, that is, a point x with $f(x) = x$.

Problem 5. Prove that there exists a space X such that $H_1(X) = \mathbb{Z}_{2021}$. Find the fundamental group of such a space.

Qualifying Exam

Math 6390.501 **Introduction to Quantum Field Theory** Spring 2021

Answer any **four** *questions, each question worth 25 points*

1. A Lorentz transformation $x^\mu \longrightarrow x^{'\mu} = L^\mu{}_\nu x^\nu$ preserves the standard Lorentz metric $g_{\mu\nu}$, so that $g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x^{'\mu}x^{'\nu}$ for all x.

(a) Show that the above equation implies

$$g_{\mu\nu} = g_{\sigma\tau}L^\sigma{}_\mu L^\tau{}_\nu$$

and that an infinitesimal Lorentz transformation has the form

$$L^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

(b) What are the symmetry properties of $\omega^\mu{}_\nu$?

(c) Show that the following the tensor field $T^{\mu\nu} = x^\mu x^\nu$ is invariant under Lorentz transformation.

(d) Given the relativistic invariance of the measure d^4p, show that the integration measure

$$\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}$$

is invariant under Lorentz transformation, provided $E_p = \sqrt{\mathbf{p}^2 + m^2}$.

2. (a) Starting from the action

$$S = \int d^4x\overline{\psi}(x)(i\,\gamma^\mu\partial_\mu - m)\psi(x)$$

where $\overline{\psi}(x) = \psi^\dagger\gamma^0$, derive Dirac's equation.

(b) Under a Lorentz transformation the Dirac wave function should transforms as follows:

$$\psi(x) \longrightarrow \psi_L(x) = S(L)\psi(L^{-1}x)$$

where $S(L)$ is a representation of Lorentz transformation acting on the spin labels of $\psi(x)$. For the Dirac equation to be Lorentz invariant $\psi_L(x)$ should also satisfies the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi_L(x) = 0.$$

Prove that this implies $\gamma^\mu\partial^{'}_\mu = S^{-1}(L)(\gamma^\mu\partial_\mu)S(L)$. Show that this is equivalent to γ matrices transforming as follows: $S^{-1}(L)\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu$.

(c) Use part (b) to prove that the action in part (a) is Lorentz invariant.

(d) Show how the axial vector current $A^\mu = \overline{\psi}\gamma^5\gamma^\mu\psi$ transforms under Lorentz transformation.

3. The energy momentum 4-vector operator in quantized Klein-Gordon field (i.e. free scalar field with Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial\phi(x))^2 - \frac{1}{2}m^2(\phi(x))^2$) can be put in the form (don't prove this)

$$P^\mu = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} p^\mu a^\dagger(p)a(p),$$

where $a(p)^\dagger$ and $a(p)$ are the mode operators in the expansion of $\phi(x)$ and obey the usual commutation relations

$$[a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0,\ [a(p), a^\dagger(p')] = (2\pi)^3 2E_p\delta^{(3)}(\mathbf{p}-\mathbf{p}').$$

Compute the commutators of P^μ with $a(p)^\dagger$ and $a(p)$ and using them explain the significance of the operators $a(p)^\dagger$ and $a(p)$.

4. The operator N is defined in quantized Klein-Gordon theory as

$$N = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} a(p)^\dagger a(p)$$

Calculate the commutator of N with $a(p)^\dagger$ and $a(p)$ and deduce that N is the particle number operator.

5. (a) The Fourier decomposition of the Dirac field operator $\psi(x)$ in the Heisenberg picture is given by

$$\psi(x) = \sum_\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \left[a(p,\lambda)u_\lambda e^{-ip.x} + b^\dagger(p,\lambda)v_\lambda e^{ip.x}\right]$$

Construct a similar expression for the canonical momentum conjugate to $\psi(x)$. Write down the non-vanishing anticommutators of the creation and annihilation operators.

(b) Show that the quantum Hamiltonian of the Dirac field can be written as

$$H = \int d^3x\overline{\psi}(x)(-i\gamma^i\partial_i + m)\psi(x)$$

6. Using the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{a} = a_\mu\gamma^\mu$, prove the following results *without using any particular representation*:

(a) $\mathrm{Tr}\gamma^\mu = 0$, (b) $\mathrm{Tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$, (c) $\mathrm{Tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0$

(d) $\not{a}_1\not{a}_2 = 2a_1.a_2 - \not{a}_2\not{a}_1 = a_1.a_2 + \frac{1}{2}[\gamma^\mu,\gamma^\nu]a_{1\mu}a_{2\nu}$, (e) $\mathrm{Tr}(\not{a}_1\not{a}_2) = 4a_1.a_2$

7. (a) Stating from the definitions, prove that (for two real scalar field operators)

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : +i\Delta_F(x-y)$$

where the symbols stand for their usual meanings.

(b) There are three terms in the above equation. Indicate which one or ones is/are operator(s) and which one or ones is/are c-number(s) or both.

(c) State (mathematically) Wick's theorem involving n number of fields and using this express $T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))$ in terms of normal ordered products and Feynman propagators.