Qualifying Exam
Math 6301 Winter 2019
Real Analysis

Name______________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Is the following statement true or false?

   "If the boundary of $\Omega \subset \mathbb{R}^n$ has outer measure zero, then Ω is measurable?"

   Justify your answer.

2. Does the sequence of functions $\{f_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ given by
   $$f_n(x) := e^{-n|1-\sin(x)|}$$
   converge in measure to $f(x) \equiv 0$? Justify your answer.

3. Compute $\int_{[0,\pi/2]} f d\mu$, where
   $$f(x) = \begin{cases} \sin^3(x), & \text{if } \cos(\text{x}) \text{ is rational;} \\ \cos^6(x), & \text{if } \cos(\text{x}) \text{ is irrational} \end{cases}$$

4. Let
   $$f_n(x) = \sqrt{n} x e^{-nx^3}, \quad x \in [0,1], n \in \mathbb{N}.$$
   Does there exist a finite limit
   $$\lim_{n\to\infty} \int_{[0,1]} f_n d\mu \quad ?$$
   Justify your answer.

University of Texas at Dallas

QUALIFTYING EXAM

Functional Analysis	
Exam Date:	**Identification Number:**
January 5, 2019	....

Instructions:

1. Use the space provided to write your solutions in this booklet to write the final (neat, elegant and precise) version of your solutions.
3. Provide all the necessary definitions and state all the theorems that you need for your solutions.

Question	Weight	Your Score	Comments
1.	25		
2.	25		
3.	25		
4.	25		
Total:	100		

Problem 1. In the space l^2, equipped with the standard norm

$$\|x\|_2 := \left(\sum_{k=1}^{\infty} x_k^2 \right)^{\frac{1}{2}}, \quad x = (x_k)_{k=1}^{\infty} \in l^2.$$

we define the function

$$\|x\|_a := \left(\sum_{k=1}^{\infty} x_k^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \frac{1}{2^k} |x_k| \right)^{\frac{1}{2}},$$

Show that the norms $\| \cdot \|_a$ is an equivalent to $\| \cdot \|_2$ norm.

Problem 2: Let $\mathbb{E}$ be a normed space and A, $B \subset \mathbb{E}$ two closed sets.

(a) Show that if one of the sets A or B is compact, then the set $A + B$ is closed,

(b) Give an example of two closed sets in a Banach space such that $A + B$ is not a closed set.

Problem 3: Let $\mathbb{E}$ be a normed space and $f : \mathbb{E} \to \mathbb{R}$ a linear functional. Suppose there exists $x_o \in \mathbb{E}$ and $\delta > 0$ such that

$$\sup_{x \in B_\delta(x_o)} f(x) < \infty.$$

Show that $f \in \mathbb{E}^*$, i.e. f is a continuous linear functional.

Problem 4. Consider the Banach space $\mathbb{E} := l^2$, For every $n \in \mathbb{N}$ we define the operator $T_m : \mathbb{E} \to \mathbb{E}$ by

$$T_m(x) = (\underbrace{0, \dots, 0}_{m-1\,\times}, x_m, x_{m+1}, \dots), \quad x = (x_1, x_2, \dots, x_m, x_{m+1}, \dots) \in l^2.$$

(a) Compute the norm $\|T_m\|$;

(b) Show that there exists $T : \mathbb{E} \to \mathbb{E}$ such that

$$\forall_{x \in \mathbb{E}} \quad \lim_{m \to \infty} T_m x = Tx.$$

(c) Show that $\{T_m\}$ doesn't contain any subsequence convergent in $L(\mathbb{E})$.

Name Code: ______________

Qualifying Exam: Ordinary Differential Equations I, January 2019

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 25 points each. Give clear and complete answers with full details in proofs.

1. Given, the initial values problem; $x' = f(x)$ with $x(0) = 0$ and

$$f(x) = \begin{cases} \sqrt[3]{x^2} & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0 \end{cases}$$

   a) Without finding a solution; justify that this IVP has a solution in any small nbhd of $(0, 0)$.
   b) Without explicitly appealing to the solution(s) of this IVP; justify uniqueness (or lack of uniqueness) of the solution for this IVP.
   c) Explicitly find the solution(s) for this IVP, and show its uniqueness or lack of it.

2. For $\mathbf{x} \in R^n$,
   (a). Prove that:

$$|\mathbf{x}|_\infty \leq |\mathbf{x}|_2 \leq \sqrt{n}|\mathbf{x}|_\infty$$

   (b). Prove that the uniform norm of matrix $A = [a_{ij}]$, where i & $j = 1, 2, ...n$ is:

$$|A|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

3. Find a fundamental set of solutions to the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= y_1 + y_2 \end{aligned}$$

   by reducing the problem to the Jordan canonical form.

4. Find all values of a and b (if any), with $0 \leq a < b \leq \pi$, that make the differential operator L defined by

$$Ly = \frac{d}{dt}[(2 + \sin t)\frac{dy}{dt}] + (\cos t)y, \quad y(a) = y(b), \quad \text{and } y'(a) = y'(b),$$

   to be self-adjoint.

Algebra Qualifying Exam

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. There is one bonus problem. Show your work and justify all statements.

Problem 1 (25 points). Let X be a set. Show that the set of subsets of X has the structure of a commutative associative ring when addition and multiplication are defined as

$$\begin{aligned} A + B &= (A \setminus B) \cup (B \setminus A) \\ A \cdot B &= A \cap B. \end{aligned}$$

Problem 2 (25 points).

(a) (10 points) Find a Sylow 3-subgroup of the alternating group A_6.

(b) (10 points) Compute the normalizer in A_6 of the Sylow 3-subgroup found in part (a).

(c) (5 points) How many Sylow 3-subgroups does A_6 have?

Problem 3 (25 points).

(a) (15 points) What is the largest possible order of an element in the symmetric group S_6?

(b) (10 points) How many elements in S_6 have exactly that order?

Problem 4 (25 points).

(a) (15 points) Determine the number of group homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$.

(b) (10 points) Determine the number of ring homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$. (For the purposes of this problem, a ring homomorphism must send 1 to 1.)

Bonus (10 points). Show that the number of units in the ring $\mathbb{Z}/n\mathbb{Z}$ is even for $n \geq 3$.

Complex Analysis Qualifying Exam

Spring 2019

January 8, 2019

1. [**25 points**] **True** or **false** (Justification is needed):

(a) There exists a fractional-linear transformation φ such that φ(0) = 0, φ(1) = 1, φ(2) = 2, and φ(3) = ∞.

(b) The function $p(z) = z^9 + z^5 - 8z^3 + 2z + 1$ has 6 roots inside the annulus $1 < |z| < 2$.

(c) If f and g are analytic on the unit disk $D = \{z \in C \mid |z| < 1\}$, and if they coincide on a non-empty set *S* which is closed in *D*, then $f = g$ in *D*.

2. [**25 points**] In parts (a), (b) and (d) *U* is an open connected subset of *C* and $f : U \to C$.

(a) What is meant by the statement that f is *complex differentiable* at $z_0 \in U$? What does it mean to say that f is *holomorphic* at $z_0 \in U$?

(b) Let u and v denote the real and imaginary parts of f. Show that if f is complex differentiable at $z_0 = x_0 + i\,y_0 \in U$ then the partial derivatives of *u* and *v* exist at (x_0, y_0) and satisfy the *Cauchy-Riemann equations* there.

(c) Let $g(z) = z(z + \bar{z})$. Find all complex points where g is complex-differentiable and compute the derivative of g there.

(d) Prove that if $f(z)$ and $f(\bar{z})$ are both holomorphic in a domain *U*, then $f(z)$ is constant in *U*.

3. [**25 points**] Prove that the set of all holomorphic bijections (injective and surjective maps) of the unit disk $D = \{z \in C \mid |z| < 1\}$ is a group and consists of all transformations of the following form:

$$f(z) = e^{i\theta}\,\frac{z - a}{1 - \bar{a}\,z},$$

where $a \in D$ and $\theta \in R$.

4. [**25 points**] In this question all curves are assumed to be positively oriented. Compute:

(a) the integral $\displaystyle\int_{|z|=1} \frac{\sin z \cdot \cos 3z}{z^4}\,dz$;

(b) the integral $\displaystyle\int_{|z|=2} \frac{f(z)}{(z-1)^2}\,dz$, where f is holomorphic on the disk $D_3 = \{z \in C \mid |z| < 3\}$ and suppose $f(1) = f'(1) = -1$;

(c) the integral $\displaystyle\int_{|z|=2} \frac{f(z)}{z^2 - 1}\,dz$, where f is holomorphic on the disk $D_3 = \{z \in C \mid |z| < 3\}$ and $f(1) = 2$ and $f(-1) = 1$.

Examination Booklet

MATH 6324: Applied Dynamical Systems I

QE Exam, Winter 2018 – 2019

There are FIVE problems on this paper.
Solve THREE problems for full grade.

QE ID: ____________________
(print)

1. Consider the overlapping generation model

$$
\begin{aligned}
P_{n+1} &= s(3P_n + Q_n)\big(1 - 0.01(P_n + Q_n)\big) \\
Q_{n+1} &= P_n
\end{aligned}
$$

Find the components of the positive fixed point $(P^*(s), Q^*(s))$ as functions of the parameter s. Find the Hopf bifurcation value s_H of the parameter. Explain the dynamical scenario in a neighborhood of the fixed point as s is varied across the bifurcation value s_H.

2. Consider a Phase Locked Loop system of the form

$$
\begin{aligned}
x' &= y \\
y' &= z \\
z' &= \cos(x) - y - az
\end{aligned}
$$

where is a parameter. This system has an equilibrium at the point $(x, y, z) = (\pi/2, 0, 0)$ for all the values of the parameter a.

Determine the Hopf bifurcation value a_H of the parameter for the equilibrium point $(\pi/2, 0, 0)$.

Investigate stability of this equilibrium for $a < a_H$ and $a > a_H$.

Assume that the small cycle born via the Hopf bifurcation coexists with the unstable equilibrium near the bifurcation value a_H of the parameter. What can you say about the period and stability of the cycle?

3. Draw the phase portrait of the nonuniform oscillator as a function of the parameter μ:

$$\dot{\theta} = \mu + \cos\theta$$

Sketch the bifurcation diagram of equilibrium points θ^* vs μ indicating the stability. Determine the type of the bifurcations. Sketch all the qualitatively different one-dimensional phase portraits as the parameter μ is varied.

4. Transform the logistic map $x_{n+1} = rx_n(1 - x_n)$ to the quadratic map $y_{n+1} = y_n^2 + c$ by a linear change of variables, $x_n = ay_n + b$, where a, b are to be determined.

   Since the change of variables is invertible, the two maps are topologically conjugate, hence they have the same dynamics. Show that the interval $-2 \le y \le 2$ is invariant for the map $y_{n+1} = y_n^2 - 2$, and this map is chaotic on this interval.

   Find the distribution function of trajectories of the map $y_{n+1} = y_n^2 - 2$ on the interval $[-2, 2]$. The distribution function of the the map $x_{n+1} = 4x_n(1 - x_n)$ is $D(z) = z$.

5. Consider the system

$$\begin{aligned} x' &= x\cos a - y\sin a + x^2 \\ y' &= x\sin a + y\cos a - xy \end{aligned}$$

with a parameter $a \in [0, 2\pi)$. Show that zero is an isolated equilibrium of this system for all a. Show that the topological index of this equilibrium is independent of a and calculate this index. For which values of a the equilibrium is stable and for which it is unstable?

The end

MATH 6319 - Ph. D Qualifying Examinations Spring 2019

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** State the law of complementary nullities. Use it to find the structure of the following submatrices $\begin{pmatrix} s_{41} & s_{42} & s_{43} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{66} \end{pmatrix}$ and $\begin{pmatrix} s_{13} & s_{14} & s_{15} & s_{16} \\ s_{23} & s_{24} & s_{25} & s_{26} \\ s_{33} & s_{34} & s_{35} & s_{36} \\ s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix}$ where S is the inverse of a 6×6 tridiagonal matrix T. You may assume that none of the entries of T on the diagonal and the first superdiagonal and subdiagonal is zero.

  (5 + 10 + 10 =25 points)

- **Q2** Compute in closed form the eigenvalues of the matrix $A = (i + j)$. Explain your work fully.

  25 points)

- **Q2** Let V be a complex pre-inner product space. Find a suitable quotient space V/W and suitable inner-product on it which is related to that on V. Justify your answer fully (i.e., identify W, explain why it is a subspace; explain why your inner product is well defined and why it is indeed an inner product etc.,)

  (25 points)

- **Q4**

  State and prove Fischer's inequality for positive semidefinite matrices.

(25 points)

Q5 Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier transform. All results concerning differentiation and the Fourier transform must be clearly and cleanly stated.

Q6 Find the convolution of f and g where both are the characteristic function of $[0, 1]$.

MATH 6310 (TOPOLOGY) – JANUARY 2019 QUALIFYING EXAM

QE ID:

There are 5 problems. Each problem is worth 20 points. The total score is 100 points.

Show all your work to get full credits.

Problem 1. The real projective n-space $\mathbb{R}P^n$ is defined as the quotient space of $\mathbb{R}^{n+1}\backslash\{0\}$ by the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$. Find a cell decomposition of $\mathbb{R}P^n$.

Problem 2. Compute the fundamental group of the quotient space of the unit sphere $S^2 \subset \mathbb{R}^3$ obtained by identifying the points $(1, 0, 0)$ and $(0, 1, 0)$.

Problem 3. Let X be the topological space obtained from a triangle by identifying its three vertices to a single point. Compute the (simplicial) homology groups of X.

Problem 4. Prove the Brouwer fixed point theorem: Every continuous map $f : D^n \to D^n$ has a fixed point, that is, a point x with $f(x) = x$.

Problem 5. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the (cellular) homology groups of X.