Qualifying Exam
Math 6301 January 2018
Real Analysis I

QE ID ____________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Let $\mathcal{M}$ be the space of classes of equivalent measurable functions on $[a,b]$. Given $f, g \in \mathcal{M}$, put

$$\rho(f,g) := \int_{[a,b]} \frac{|f-g|}{1+|f-g|} d\mu,$$

where "$\int_{[a,b]}$" stands for the Lebesgue integral.

(i) Show that ρ is a metric.

(ii) Is $(\mathcal{M}, \rho)$ a complete metric space? Justify your answer.

(iii) Does the convergence in $(\mathcal{M}, \rho)$ coincide with the convergence in measure? Justify your answer.

2. Let $A \subset \mathbb{R}^2$ be the graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \in [-1,1] \setminus \{0\}; \\ 2018, & \text{if } x = 0. \end{cases}$$

Is the set $A \cup \{(0,y) \in \mathbb{R}^2 \ : \ -1 \le y \le 1\}$ measurable with respect to the Lebesgue measure on the plane? Justify your answer.

3. (i) Show that

$$\int_{[0,1]} \Big(\int_{[0,1]} \frac{x^2-y^2}{(x^2+y^2)^2} dy \Big) dx = \frac{\pi}{4} \text{ and } \int_{[0,1]} \Big(\int_{[0,1]} \frac{x^2-y^2}{(x^2+y^2)^2} dx \Big) dy = -\frac{\pi}{4}.$$

(ii) Does this contradict Fubini's theorem? Justify your answer.

4. Let $\{g_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of integrable functions convergent a.e. to g. Suppose, in addition, that g_n is non-negative a.e. for any $n \in \mathbb{N}$.

(i) Is g integrable? Justify your answer

(ii) Is the following statement true:

$$\int_{[0,1]} g_n e^{-g_n} d\mu \to \int_{[0,1]} g e^{-g} d\mu.$$

Justify your answer.

Good luck!

Functional Analysis I

Qualifying Exam, January 2018

Write your QE ID number (given to you by Angie) on all answer sheets.

Do NOT put your name, UTD ID, or any other identifying information on any of your answer sheets.

Solve all four problems in this package.

Question	**Weight**	**Your Score**	**Choice**
1.	25		
2.	25		
3.	25		
4.	25		
Total:	**100**		

Problem 1. Let $\mathbb{E}$ be a Banach space and $\varphi : \mathbb{E} \to (-\infty, \infty]$ a function such that $D(\varphi) \neq \emptyset$. Define the so-called *conjugate function* $\varphi^* : \mathbb{E}^* \to (-\infty, \infty]$ by

$$\forall_{f\in\mathbb{E}^*} \;\; \varphi^*(f) := \sup x \in \mathbb{E}\Big(\langle f, x\rangle - \varphi(x)\Big)$$

(a) Show that φ^* is a convex function.

(b) Show that φ^* is a lower semi-continuous function.

(c) Consider the Euclidean space $\mathbb{E} = \mathbb{R}^n$ and $\varphi : \mathbb{E} \to \mathbb{R}$ given by

$$\forall_{x\in\mathbb{E}} \;\; \varphi(x) = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \quad x = (x_1, x_2, \dots, x_n),\; p \geq 1.$$

Find φ^*.

Problem 2. Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces and $A : \mathbb{E} \to \mathbb{F}$ a linear operator.

(a) Show that A is continuous if and only if

$$\sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} < \infty.$$

(b) Show that for continious linear operators $A : \mathbb{E} \to \mathbb{F}$ the function $\|A\| := \sup_{\|x\| \leq 1} \|Ax\|$ satisfies the conditions of a norm.

(c) Suppose that $\mathbb{E} = \mathbb{F} = C([0, 2\pi]; \mathbb{R})$ is the space of all continuous function $\varphi : [0, 2\pi] \to \mathbb{R}$ equipped with the standard sup-norm

$$\|\varphi\|_\infty := \sup_{t \in [0, 2\pi]} |\varphi(t)|, \quad \varphi \in C([0, 2\pi]; \mathbb{R}),$$

and assume that $(A\varphi)(t) := \sin(t)\varphi(t)$. Compute $\|A\|$.

Problem 3. Assume that $\mathbb{E}$ is a Banach space and $X \subset \mathbb{E}$.

(a) Explain when the set X is compact in $\mathbb{E}$ (provide a definition and other conditions for compactness)

(b) Show that the closed unit ball $B := \{x \in \mathbb{E} : \|x\| \leq 1\}$ in $\mathbb{E}$ is compact if and only if $\mathbb{E}$ is finite-dimensional.

Problem 4. Let $\mathbb{E}$ be a Banach space.

(a) Explain when a sequence $\{x_n\} \subset \mathbb{E}$ is weakly convergent to $x_o \in \mathbb{E}$ (provide a definition and other facts related to weak convergence)

(b) Give an example of a Banach space $\mathbb{E}$ and a sequence $\{x_n\} \subset \mathbb{E}$ that is weakly convergent but not convergent in the usual (i.e. strong) topology.

(c) Show that every weakly convergent sequence $\{x_n\} \subset \mathbb{E}$ is bounded in $\mathbb{E}$.

Complex Analysis Qualifying Exam

Spring 2018

January 8, 2018

1. [25 points] **True** or **false** (Justification is needed):
   a) If $u(x, y)$ and $v(x, y)$ are harmonic in a domain D then the product $u(x,y)\cdot v(x, y)$ is also harmonic in D.
   b) The function $p(z) = z^5 + iz^3 - 4z + i$ has 4 roots inside the annulus $1 < |z| < 2$.
   c) If f is continuous on a domain D, then f must be complex differentiable at least at one point in D.
2. [25 points] Let $f(z) = |z+1|^2$. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ be the path that describes the unit circle with center 0 anticlockwise.
   (a) Show that f is not holomorphic at any point of $\boldsymbol{C}$.
   (b) Find a function g that is holomorphic on some domain that contains γ and such that $f(z) = g(z)$ at all points on the unit circle γ. (It follows that $\int_\gamma f = \int_\gamma g$.)
   (c) Hence, calculate $\int_\gamma |z+1|^2 \, dz$.
3. [25 points]
   (a) State Rouche's Theorem.
   (b) State Schwarz's Lemma.
   (c) Suppose f is holomorphic in the unit disk $|z| < 1$ with $f(0) = 0$ and $|f(z)| \le 1$. Prove that for any integer $m \ge 1$ the function $f(z) - 2^m z^m$ has precisely m zeros counting multiplicity in the open disk $|z| < ½$.
4. [25 points] Compute
   (a) the integral $\int_0^\pi \frac{dt}{5 + 3\cos t}$ using complex methods;
   (b) the residue at $z = 0$ of the function $f(z) = \frac{e^{2z^2} - 1}{z^3}$;
   (c) the residue at $z = \infty$ of the function $f(z) = z^n e^{10/z}$, $n \in N$.

Abstract Algebra Qualifying Exam

January 2018

QE ID________________________

Instructions. *Please solve any four problems from the list of the following problems*

1. Prove that a subgroup of index 2 inside a finite group is normal.
2. Solve the following problems:

a) Prove that the order of an element in S_n equals the least common multiple of the lengths of the cycles in its cycle decomposition.

b) Show that the number of m-cycles in S_n is given by

$$\frac{n\,(n-1)\,(n-2)\ldots(n-m+1)}{m}.$$

3. Find an isomorphism
$$\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/\gcd(a,b)\mathbb{Z} \times \mathbb{Z}/\operatorname{lcm}(a,b)\mathbb{Z}.$$
4. Suppose that a group G of order 12 acts transitively on a set X. What are the possible cardinalities of X?
5. Show that the symmetric group of order $n!$ is generated by the elements $(1,2)$ and $(1,2,\ldots,n)$.
6. Prove that there are no simple groups of order 56.

Qualifying Exam: Ordinary Differential Equations I, January 2018

QE ID: ______________

THIS IS A CLOSED BOOK, CLOSED NOTES & NO CALCULATOR EXAM

Problems count 25 points each

Give clear and complete answers with full details in proofs

1. Prove that if $f \epsilon C(D)$ and if f satisfies a Lipshitz condition in D with Lipschitz constant L, then the initial value problem $x' = f(t, x)$ and $x(\tau) = \xi$, with $(\tau, \xi) \in D$ has at most one solution on any interval $|t - \tau| \leq d$. Note: D is an open, connected, nonempty subset of R^2 and $(t, x) \in D$.

2. Prove that if $\Phi(t)$ is a fundamental set of solutions of $x' = A(t)x$ with periodic coefficient $A(t) = A(t+T)$, then $\Phi(t+T)$ is also a fundamental set of solutions of $x' = A(t)x$. Furthermore exist a non-singular periodic matrix $P(t)$, with period T and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$.

3. Suppose that for a continuous function $f(t)$ we are given that the equation

$$x' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix} x + f(t)$$

has at least one solution $\phi_p(t)$ which satisfies

$$\sup\{|\phi(t)| : \tau \leq t < \infty\} < \infty.$$

Show that all the solutions of above ODE satisfy this boundedness condition.

4. For what values of a and b (if any) with $0 \leq a < b \leq \pi$, is the differential operator L defined by

$$Ly = \frac{d}{dt}[(2 + \sin t)\frac{dy}{dt}] + (\cos t)y, \qquad y(a) = y(b), \quad \text{and} \quad y'(a) = y'(b),$$

self-adjoint?

ID NUMBER:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 4 of the 5 problems to solve. Please indicate which 4 problems you would like graded.

(1) If $e^{1.3}$ is approximated by Lagrangian interpolation from the values $e^0 = 1$, $e^1 = 2.7183$, and $e^2 = 7.3891$, give an upper bound for the error. Compare to the actual error.

(2) Derive the following two formulas for approximating the first derivative. Find their error terms. Which formula is more accurate?

$$(a) f'(x) \approx \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)]$$

$$(b) f'(x) \approx \frac{1}{2h}[-3f(x) + 4f(x+h) - f(x+2h)]$$

(3) Prove that the kth divided difference of a polynomial of degree $< k$ is 0.

(4) Following IEEE standards, we can design a hypothetical computer with 32 bit storage. This machine represents a nonzero real number in binary using 1 bit for the sign of the real number, 8 bits for storing the exponent, and the remaining 23 bits for the mantissa. Prove that 4/5 can not be represented exactly on this machine. What is the closest binary number that fits exactly on this machine? What is the relative roundoff associated with storing this number on the hypothetical 32-bit machine?

(5) The Fourier series representation of a function $f(x)$ (assuming suitable smoothness of f so the series converges) is defined by

$$f(x) = \sum_{k=0}^{\infty} (A_k \cos k_x + B_k \sin k_x)$$

with coefficients

$$A_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos k_x dx,$$

and

$$B_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin k_x dx.$$

Show that the coefficient B_k is the best possible choice in the least squares (or best approximation) sense of a trigonometric interpolating polynomial. (Note that the proof for A_k is similar so you need not prove it. Also,

$$\sin^2 x = \frac{1 - \cos 2x}{2}.)$$

MATH 6319 - Ph. D Qualifying Examinations
Spring 2018

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** State the law of complementary nullities. Use it to find the structure of the following submatrices $\begin{pmatrix} s_{41} & s_{42} & s_{43} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{66} \end{pmatrix}$ and $\begin{pmatrix} s_{13} & s_{14} & s_{15} & s_{16} \\ s_{23} & s_{24} & s_{25} & s_{26} \\ s_{33} & s_{34} & s_{35} & s_{36} \\ s_{63} & s_{64} & s_{65} & s_{66} \end{pmatrix}$ where S is the inverse of a 6×6 tridiagonal matrix T. You may assume that none of the entries of T on the diagonal and the first superdiagonal and subdiagonal is zero.

(5 + 10 + 10 =25 points)

- **Q2** Compute in closed form the eigenvalues of the matrix $A = (i - j)$. Explain your work fully.

25 points)

- **Q2** State carefully the term recurrence relations satisfied by orthonormal polynomials. Reformulate this recurrence in terms of matrices and explain why this implies that the zeroes of such polynomials are real and simple.

(25 points)

- **Q4**

State and prove Fischer's inequality for positive semidefinite matrices.

(25 points)

Q5 Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier transform. All results concerning differentiation and the Fourier transform must be clearly and cleanly stated.

Q6 Let P and Q be projections. i) State a necessary and sufficient condition for $P+Q$ to be a projection. State also what subspace $P+Q$ projects onto and along which subspace. ii) Use this to find (i.e, with proof) a necessary and sufficient condition for $P-Q$ to be a projection. iii) State and prove also what subspace $P-Q$ projects onto and along which subspace.

(6 + 9 + 10 = 25 points)

Q7 Let $A_{n\times n}$ be invertible. Find (i.e., with justification) the closest (in the spectral norm) singular matrix to A.

(25 points)