Qualifying Exam
Math 6301 Spring 2017
Real Analysis

Name______________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Given $A \subset [0,1]$, take a function
$$f(x) := \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A \end{cases}$$
Does there exist a differentiable function $g : [0,1] \to \mathbb{R}$ such that $g'(x) = f(x)$ for all $x \in [0,1]$? Justify your answer.

2. Let $A \subset \mathbb{R}^2$ be the graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by:
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } \ x \in [-1,1] \setminus \{0\}; \\ 2017, & \text{if } \ x = 0. \end{cases}$$
Is the set $A \cup \{(0,y) \in \mathbb{R}^2 \ : \ -1 \leq y \leq 1\}$ measurable with respect to the Lebesgue measure on the plane? Justify your answer.

3. (a) Let
$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset I_0 =: [a,b],$$
be a sequence of measurable sets and let $f : [a,b] \to \mathbb{R}$ be a non-negative integrable function. Show that
$$\lim_{n\to\infty} \int_{E_n} f(x)d\mu = \int_E f(x)d\mu \quad \text{where} \quad E = \bigcup_{n=1}^{\infty} E_n.$$
(b) Is the above statement true without the hypothesis "f is non-negative"? Justify your answer.

4. Compute $\int_{[0,\pi/2]} f d\mu$, where
$$f(x) = \begin{cases} \cos^3(x), & \text{if } \ \sin(\mathrm{x}) \ \text{ is rational;} \\ \sin^3(x), & \text{if } \ \sin(\mathrm{x}) \ \text{ is irrational} \end{cases}$$

University of Texas at Dallas

Functional Analysis I

Qualifying Exam, Spring 2017

Instructions

(a) Attempt to solve all four questions, however only the best three of your solutions will be considered.

(b) Include all the related to the problems definitions and state all the results that you are using in your solutions.

(c) Please write clearly and use your notation carefully, so we can avoid unnecessary misunderstandings.

Problem 1. Let $\mathbb{E}$ be a Banach space and $L \subset \mathbb{E}$ a linear subspace. Show that

$$\forall_{f_o \in \mathbb{E}^*} \, \exists_{g_o \in L^\perp} \quad \operatorname{dist}(f_o, L^\perp) = \|f_o - g_o\|.$$

Problem 2. Let $\mathbb{E}$ be a Banach space and $A \subset \mathbb{E}$ such that A is compact with respect to weak topology $\sigma(\mathbb{E}, \mathbb{E}^*)$. Show that A is bounded.

Problem 3. Consider the space $\mathbb{R}^n$. For $p \geq 1$ we define the following norm on $\mathbb{R}^n$

$$\|x\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \quad x = (x_1, x_2, \ldots, x_k)^T \in \mathbb{R}^n,$$

and

$$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}, \quad x = (x_1, x_2, \ldots, x_k)^T \in \mathbb{R}^n,$$

. Show that

(a) For every $x \in \mathbb{R}^n$ we have $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$;

(b) For every $x \in \mathbb{R}^n$ we have $\|x\|_p \leq n^{\frac{q-p}{pq}} \|x\|_q$, $1 \leq p \leq q$;

(c) For every $x \in \mathbb{R}^n$ we have $\|x\|_q \leq \|x\|_p$, $1 \leq p \leq q$;

Problem 4. Let $\mathbb{E}$ be a Banach space and C, $D \subset \mathbb{E}$ two convex sets. Show that

(a) $\operatorname{conv}(C \cup D) = \bigcup_{t \in [0,1]} tC \cup (1-t)D$;

(b) the interior $\operatorname{int}(C)$ of the set C is a convex set;

(c) the closure $\overline{C}$ of the set C is a convex set.

Complex Analysis Qualifying Exam

Spring 2017

April 7, 2017

Definition. Given a map $f : A \to A$. An element $b \in A$ is a *fixed point* of f, if $f(b) = b$.

1. All maps considered here are Mobius transformations of the Riemann sphere to itself. We say that T is conjugate to T_0 if there exists a transformation S such that $S^{-1}TS = T_0$, that is, T maps Sz to Sw whenever T_0 maps z to w. Show that
   (a) A dilation $z \mapsto \lambda z$, $\lambda \notin \{0, 1\}$ has two fixed points, while a translation $z \mapsto z + c$, $c \neq 0$ has one fixed point.
   (b) If T has exactly two fixed points, then T is conjugate to a dilation $z \mapsto \lambda z$, $\lambda \notin \{0, 1\}$.
   (c) If T has exactly one fixed point, then T is conjugate to the translation $z \mapsto z + 1$.

2. For $z \neq 0$ let
$$f(z) = (z-6) \cdot e^{\frac{1}{z}} \ .$$
   (a) Find the Laurent expansion of f at $z = 0$.
   (b) Where does the Laurent series converge?
   (c) What type of isolated singularity does f have at $z = 0$?
   (d) What is the residue of f at $z = 0$?
   (e) Calculate $\displaystyle\int_{|z+6|=2} z \cdot e^{\frac{1}{z+6}} \, dz$.

3. (a) How many roots of the polynomial $z^7 - 5z^3 + 12$ lie in the region $A = \{z \mid 1 < |z| < 2\}$? How many roots of this polynomial lie in the complement of A?

   (b) Suppose f is analytic in the closed disk $\overline{D}(0, 1) = \{z : |z| \leq 1\}$ and $|f(z)| < 1$ for all $z \in \overline{D}(0, 1)$. Prove that f has unique fixed point in $D(0, 1)$.

4. Prove that a non-constant entire function maps $\boldsymbol{C}$ onto a dense subset of $\boldsymbol{C}$.

Abstract Algebra Qualifying Exam

April 5, 2017

Name__________________________

Instructions. *Please solve any four problems from the list of the following problems (show all your work).*

1. Let S_4 be the symmetric group of all permutations of a set of 4 elements.

a) List all the elements of S_4 arranged in the conjugacy classes.

b) Find a subgroups of S_4 that is isomorphic to the symmetric group S_3 of all permutations of a set of 3 elements.

c) Find a subgroup of S_4 that is isomorphic to the dihedral group D_4 of symmetries of a square.

2. Let G be a group and H be a subgroup of G. Denote by $N_G(H)$ the normalizer of H in G and by $C_G(H)$ the centralizer of H in G.

a) Show that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the group $\mathsf{Aut}(H)$ of all automorphisms of H.

b) Let D_3 denotes the dihedral group of all rigid motions of an equilateral triangle. Find the group $\mathsf{Aut}(D_3)$ of all automorphisms of D_3 and the group $\mathsf{Inn}(D_3)$ of all inner automorphisms of D_3.

3. Let G be a nontrivial finite p-group, where p is prime.

a) Show that G has nontrivial center, i.e. $Z(G) \neq \{1\}$.

b) Show that if G has order p^2 then G is an abelian group.

4. Let G be a group and H and K be subgroups of G.

a) Show that if H and K are normal subgroups of a group G and $H \cap K = \{1\}$ then $H \leq C_G(K)$, where $C_G(K)$ denotes the centralizer of K in G.

b) Show that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = \{1\}$.

5. Let G be a group. Show that:

a) If, for all $g \in G$, $g \neq 1$, the order of g is two (i.e. $|g| = 2$) then G is an abelian group.

b) If the order of G is 36 then G is not simple.

6. Prove the following statements:

a) No group G of order 132 can be simple.

b) A group G of order 105 has a normal Sylow 5 -subgroup and a normal Sylow 7 - subgroup.

Qualifying Exam: Ordinary Differential Equations I, April 2017

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 25 points each. To receive full credit, you need to justify all your statements.

1. For $x \in \mathbf{R}$, consider the IVP $\frac{d}{dt}x = -x + g(t),\ x(0) = \xi$, where g(t) is continuous and $|g(t)| \leq 1$ for all t. Show that the IVP has a unique and bounded solution for all $t \geq 0$.

2. Consider the initial value problem $\frac{d}{dt}x = x^{1/3}$ and $x(0) = 0$ for $t \geq 0$. Show that the problem has infinitely many different solutions.

3. Consider the boundary value problem on $[0, 1]$ for the equation

$$i\frac{dx}{dt} + \lambda x = 0,$$

with $x(1) = \alpha x(0)$, where $i = \sqrt{-1}$ and α is a complex number.

   a. Find the eigenvalues

   b. Find the corresponding eigenfunctions.

   c. Find a general condition on α that will make the eignevalues to be all real.

4. Find the fundamental set of solutions to the system $\frac{d}{dt}x = Ax$, with:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Numerical Analysis Qualifying Exam April 2017

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 4 of the 5 problems to solve. Please indicate which 4 problems you would like graded.

(1) Let $x = 2^{12} + 2^{-12}$
(a) Find the machine numbers x_- and x_+ in the Marc-32 just to the left and right of x respectively. (Hint: the Marc-32 is a hypothetical 32-bit computer that follows IEEE standards. This machine represents a nonzero real number in binary using 1 bit for the sign of the real number, 8 bits for storing the exponent, and the remaining 23 bits for the mantissa.)

(b) Determine the relative error between x and $fl(x)$ (the floating point representation of x).

(2a) Estimate $f(2.14)$ using the Newton interpolating polynomial of degree 2 for this dataset:

x	2	2.1	2.2
y	1.414214	1.449138	1.483240

(b) If you are now told that the function being interpolated is $f(x) = \sqrt{x}$, using the interpolation error formula, give a tight upper bound on the error in your approximation in Part (a).

(3a) Derive the numerical differentiation formula

$$f''(x) \approx \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$$

(b) What is the order of convergence for the formula given in Part (a) above?

(4) Prove the following theorem:
Let $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal system in an inner-product space E. The best approximation of f by an element $\sum_{i=1}^{n} c_i g_i$ is obtained if and only if $c_i =< f, g_i >$.

(5) Prove that if r is a double zero of the function f (in other words, $f(r) = f'(r) = 0 \neq f''(r)$) and if f'' is continuous, then Newton's method exhibits linear convergence. (Specifically show that the error at iteration $n+1$ obeys the following rule $e_{n+1} \approx \frac{1}{2} e_n$, where $e_n = x_n - r$.)

Examination Booklet

MATH 6324: Applied Dynamical Systems

QE Exam 2017

Solve <u>FOUR</u> problems for full grade.

Last Name: ____________________ (print)

First Name: ____________________ (print)

Signature: ____________________

1. Consider the system

$$\begin{aligned}\dot{x} &= x^3 - 3xy^2 \\ \dot{y} &= 3x^2y - y^3\end{aligned}$$

It may be useful to notice that the right hand sides satisfy

$$x^3 - 3xy^2 + i(3x^2y - y^3) = (x + iy)^3,$$

hence the system can be written in complex notation as $\dot{z} = z^3$ where $z = x + iy$.

(i) Find the equilibrium point.

(ii) Determine the index of the equilibrium point.

2. Consider the system

$$\dot{x} = a + x^2$$
$$\dot{y} = y$$

with a parameter a that varies from $-\infty$ to ∞.

(i) Find and classify all the fixed points depending on a.

(ii) Show that the sum of indices of all the fixed points is conserved as a varies.

(iii) Find the bifurcation point, determine the type of the bifurcation, and sketch the bifurcation diagram.

3. Consider Schnackenberg's model of a chemical oscillator:

$$\dot{x} = a - x + x^2 y$$
$$\dot{y} = b - x^2 y$$

with parameters $a, b > 0$.

(i) Find the equation $f(a, b) = 0$ of a curve in the parameters plane (a, b) along which the necessary condition for the Hopf bifurcation is satisfied (that is, the system undergoes the Hopf bifurcation on this curve).

(ii) Sketch the Hopf curve you found in (i) on the (a, b) plane.

4. Consider the equation $\dot{x} + x = f(t)$ where f is a smooth T-periodic function.

   Is it true that this equation necessarily has a stable periodic solution $x(t)$? If yes, prove it. If not, find an f that provides a counterexample.

5. Transform the the logistic map $x_{n+1} = rx_n(1-x_n)$ to the quadratic map $y_{n+1} = y_n^2 + c$ by a linear change of variables, $x_n = ay_n + b$, where a, b are to be determined.

   (i) Since the change of variables is invertible, the two maps are topologically conjugate, hence they have the same dynamics. Show that the interval $-2 \le y \le 2$ is invariant for the map $y_{n+1} = y_n^2 - 2$, and this map is chaotic on this interval.

   (ii) Find the distribution function of trajectories of the map $y_{n+1} = y_n^2 - 2$ on the interval $[-2, 2]$.

The end

MATH 6309 (SPRING 2017) – QUALIFYING EXAM

Name:

There are 4 problems. Each problem is worth 25 points. The total score is 100 points.

Show all your work to get full credits.

Problem 1. Construct 2 different atlases for the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$.

Problem 2. Show that the real projective space $\mathbb{R}P^n$ is a smooth manifold.

Problem 3. Show that the map $f : S^n \to \mathbb{R}P^n$ defined by

$$f(x_1, \dots, x_{n+1}) = [(x_1, \dots, x_{n+1})]$$

is smooth and has constant rank n.

Problem 4. Given the following 1-forms on $\mathbb{R}^3$:

$$\begin{aligned}\theta &= -\frac{4z}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2x}{x^2+1}dz,\\ \omega &= -\frac{4xz}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2}{x^2+1}dz.\end{aligned}$$

(i) Let $\gamma(t) = (t, t, t)$, $t \in [0, 1]$. Compute $\int_\gamma \theta$ and $\int_\gamma \omega$.

(ii) Let γ be a piecewise smooth curve going from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$. Compute the above integrals.

(iii) Which of the 1-forms θ and ω is exact.

MATH 6319 - Ph. D Qualifying Examinations
Spring 2017
V. Ramakrishna

CHOICE: Do any **5** of the Qs below. Each **Q** is worth 20 points.

- **Q1** State carefully the term recurrence relations satisfied by orthonormal polynomials. Reformulate this recurrence in terms of matrices and explain why this implies that the zeroes of such polynomials are real and simple.

Q2 State the Frobenius rank inequality, and use it show the following:

- $\mathrm{r}k(XY) \leq \mathrm{m}in\ \{\mathrm{r}k(X), \mathrm{r}k(Y)\}$.
- If A is $m \times n$ and C is $n \times l$, then $\mathrm{r}k(A) + \mathrm{r}k(C) \leq n + \mathrm{r}k(AC)$.

- **Q3**

  - i) Let $C \geq D$, with C and D themselves positive semidefinite. Show that $\mathrm{d}et(C) \geq \mathrm{d}et(D)$.
  - ii) Let X be a 2×2 block matrix which is positive semidefinite. Assume the SE block, D, is invertible. Prove *Fischer's* inequality: $\mathrm{d}et(X) \leq \mathrm{d}et(A)\mathrm{d}et(D)$, where A is the NW block of X.

- **Q4** Let X and Y be $n \times n$ matrices. Prove that the characteristic polynomial of a $2n \times 2n$ matrix similar to the matrix $U = \begin{pmatrix} 0_n & X \\ Y & 0_n \end{pmatrix}$ is an even polynomial.

- **Q 5** Let A and B be real matrices which are similar. Show that the similarity can also be chosen to be real.

- **Q6** State and prove the Crabtree-Haynsowrth formula. You may assume a certain formula (which, however, must be stated precisely at the point of usage) regarding Schur complements in a product of 3 matrices.

•**Q7** Show that the Gram matrix of a finite set of vectors is always psd and that it is positive definite iff this set is linearly independent.

Q8 Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier tarnsform.

Q9 Let A be $n \times n$ <u>unreduced</u> upper Hessenberg. Show that all its eigenspaces are one dimensional. Explain further how one may <u>constructively</u> find an eigenvector corresponding to an eigenvalue.

Q10 Let f and g equal the characteristic function of $[0, 1]$. Compute in closed form their convolution.

Q11 Let $(X, <, >)$ be an inner product space. Show that the Gram matrix of a finite subset of X is psd and that it is positive definite iff this set is linearly independent.

Q12 Compute in closed form the eigenvalues of $(i - j)$.

MATH 6320 - Ph. d Qualifying Examinations
Spring 2017
V. Ramakrishna

Answer any 5 of the questions below

Q1 Show that A_{MPI} is the matrix X which minimizes $|| \; I - AX \; ||_F$.

Q2 Let C be the commutator of two square matrices A and B. Show that if C commutes with either matrix then C is nilpotent. Use this to show that a matrix A is normal iff it commutes with $C = [A, A^*]$.

Q3 Let $A \in M(n, C)$ have rank k. Show that $k + 1$ is an upper bound for the degree of the minimal polynomial of A. (Hint: Use a full rank factorization of $A = BC$ and show $BCp(BC) = Bp(CB)C$ for any polynomial p).

Q4 Let A be a matrix such that $\mathrm{T}r((A^*A)^2) = \mathrm{T}r((A^*)^2A^2)$. Show that A is normal. (The converse **need not** be shown).

Q5 Show that a matrix is nonderogatory iff every matrix that commutes with it is a polynomial in it.

Q6 Derive the JCF of a rank one matrix.

Q7 Let A have JCF $J = J_2(0) \oplus J_2(0) \oplus J_1(0) \oplus J_5(2 - i) \oplus J_2(i)$. Answer the following questions with explanations:

- What are Weyr characteristics of 0 and i?
- Construct an explicit permutation similarity between the JCF and WCF of A

Q8 Show that the S-N decomposition of a matrix is unique.

Q9 State eight conditions equivalent to the orthosymetry of a non-degenerate bilinear/sesquilinear form.

Qualifying Exam

Math 6390.501 **Introduction to Quantum Field Theory** Spring 2017

Answer any **four** *questions, each question worth 25 points*

1. A Lorentz transformation $x^\mu \longrightarrow x^{'\mu} = L^\mu{}_\nu x^\nu$ preserves the standard Lorentz metric $g_{\mu\nu}$, so that $g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x^{'\mu}x^{'\nu}$ for all x.

(a) Show that the above equation implies

$$g_{\mu\nu} = g_{\sigma\tau}L^\sigma{}_\mu L^\tau{}_\nu$$

and that an infinitesimal Lorentz transformation has the form

$$L^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

(b) What are the symmetry properties of $\omega^\mu{}_\nu$?

(c) Show that the following the tensor field $T^{\mu\nu} = x^\mu x^\nu$ is invariant under Lorentz transformation.

(d) Given the relativistic invariance of the measure d^4p, show that the integration measure

$$\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}$$

is invariant under Lorentz transformation, provided $E_p = \sqrt{\mathbf{p}^2 + m^2}$.

2. (a) Starting from the action

$$S = \int d^4x\overline{\psi}(x)(i\,\gamma^\mu\partial_\mu - m)\psi(x)$$

where $\overline{\psi}(x) = \psi^\dagger\gamma^0$, derive Dirac's equation.

(b) Under a Lorentz transformation the Dirac wave function should transforms as follows:

$$\psi(x) \longrightarrow \psi_L(x) = S(L)\psi(L^{-1}x)$$

where $S(L)$ is a representation of Lorentz transformation acting on the spin labels of $\psi(x)$. For the Dirac equation to be Lorentz invariant $\psi_L(x)$ should also satisfies the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi_L(x) = 0.$$

Prove that this implies $\gamma^\mu\partial^{'}_\mu = S^{-1}(L)(\gamma^\mu\partial_\mu)S(L)$. Show that this is equivalent to γ matrices transforming as follows: $S^{-1}(L)\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu$.

(c) Use part (b) to prove that the action in part (a) is Lorentz invariant.

(d) Show how the axial vector current $A^\mu = \overline{\psi}\gamma^5\gamma^\mu\psi$ transforms under Lorentz transformation.

3. The energy momentum 4-vector operator in quantized Klein-Gordon field (i.e. free scalar field with Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial\phi(x))^2 - \frac{1}{2}m^2(\phi(x))^2)$ can be put in the form (don't prove this)

$$P^\mu = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} p^\mu a^\dagger(p)a(p),$$

where $a(p)^\dagger$ and $a(p)$ are the mode operators in the expansion of $\phi(x)$ and obey the usual commutation relations

$$[a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0, \ [a(p), a^\dagger(p')] = (2\pi)^3 2E_p \delta^{(3)}(\mathbf{p} - \mathbf{p}').$$

Compute the commutators of P^μ with $a(p)^\dagger$ and $a(p)$ and using them explain the significance of the operators $a(p)^\dagger$ and $a(p)$.

4. The operator N is defined in quantized Klein-Gordon theory as

$$N = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} a(p)^\dagger a(p)$$

Calculate the commutator of N with $a(p)^\dagger$ and $a(p)$ and deduce that N is the particle number operator.

5. (a) The Fourier decomposition of the Dirac field operator $\psi(x)$ in the Heisenberg picture is given by

$$\psi(x) = \sum_\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \left[a(p,\lambda)u_\lambda e^{-ip.x} + b^\dagger(p,\lambda)v_\lambda e^{ip.x}\right]$$

Construct a similar expression for the canonical momentum conjugate to $\psi(x)$. Write down the non-vanishing anticommutators of the creation and annihilation operators.

(b) Show that the quantum Hamiltonian of the Dirac field can be written as

$$H = \int d^3x \overline{\psi}(x)(-i\gamma^i\partial_i + m)\psi(x)$$

6. (a) Stating from the definitions, prove that (for two real scalar field operators)

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : +i\Delta_F(x-y)$$

where the symbols stand for their usual meanings.

(b) There are three terms in the above equation. Indicate which one or ones is/are operator(s) and which one or ones is/are c-number(s) or both.

(c) State (mathematically) Wick's theorem involving n number of fields and using this express $T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))$ in terms of normal ordered products and Feynman propagators.