



University of Texas at Dallas

**Real Analysis**  
Qualifying Exam, Spring 2016

Solve all the problems in this booklet.

Question	Weight	Your Score	Choice
1.	25		
2.	25		
3.	25		
4.	25		
Total:	<b>100</b>		

**Problem 1.** Let  $\mathcal{S}$  be a  $\sigma$ -algebra on a set  $X$  and  $f, g : X \rightarrow [-\infty, \infty]$  two  $\mathcal{S}$ -measurable functions. Show that  $f + g : X \rightarrow \mathbb{R}$  is also  $\mathcal{S}$ -measurable.

**Problem 2.** Let  $(X, \mathcal{S}, \mu)$  be a measure space and  $\{F_n\}_{n=1}^{\infty} \subset \mathcal{S}$  a decreasing sequence of measurable sets such that  $\mu(F_m) < \infty$  for some  $m \in \mathbb{N}$ . Use the properties of the  $\sigma$ -algebra  $\mathcal{S}$  to show that

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n).$$

### Problem 3.

- (a) Give a definition of an outer measure  $\mu^*$  on  $X$  (for an arbitrary set  $X$ ).
- (b) Given an outer measure  $\mu^*$  on  $X$ , give a definition of a  $\mu^*$ -measurable set  $F \subset X$ .
- (c) Use the properties of the outer measure  $\mu^*$  on  $\mathbb{R}$ , given by

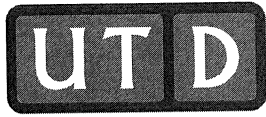
$$\mu^*(A) := \inf \left\{ \sum_{k=1}^{\infty} |\mathcal{I}_k| : A \subset \bigcup_{k=1}^{\infty} \mathcal{I}_k, \text{ where } \mathcal{I}_k \text{ are open intervals in } \mathbb{R} \right\}, \quad A \subset \mathbb{R},$$

to show that any finite interval  $\mathcal{I} \subset \mathbb{R}$  is  $\mu^*$ -measurable.

**Problem 4.** Let  $g_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of functions of bounded variations such that

$$\exists N > 0 \forall n \in \mathbb{N} \quad \text{Var}_{x \in [a, b]}(g_n) \leq N$$

and the limit  $\lim_{n \rightarrow \infty} g_n(x) = g(x)$  exists for all  $x \in [a, b]$ . Show that  $g$  is of bounded variation.



University of Texas at Dallas

# Functional Analysis I

Qualifying Exam, Spring 2016

Solve all the problems in this booklet.

Question	Weight	Your Score	Choice
1.	25		
2.	25		
3.	25		
4.	25		
Total:	<b>100</b>		



**Problem 1.** Let  $\mathbb{E}$  be a normed vector space and let  $C \subset \mathbb{E}$  be a convex set. Prove that  $\text{int}(C)$  is a convex set. Show that if  $\text{int}(C) \neq \emptyset$  then  $\overline{C} = \overline{\text{int}(C)}$ .

**Problem 2.** Let  $\mathbb{E}$  be a Banach space and  $T : \mathbb{E} \rightarrow \mathbb{E}^*$  be a linear operator such that

$$\forall x \in \mathbb{E} \quad \langle Tx, x \rangle \geq 0. \tag{1}$$

Use closed graph theorem to show that  $T$  is a bounded linear operator.

**Problem 3.** Let  $\mathbb{E}$  and  $\mathbb{F}$  be Banach spaces. Show that if  $\mathbb{E}$  and  $\mathbb{F}$  are isomorphic, then  $\mathbb{E}^*$  and  $\mathbb{F}^*$  are also isomorphic.

**Problem 4.** Let  $\mathbb{E}$  be a normed vector space and  $\varphi : \mathbb{E} \rightarrow (-\infty, +\infty]$  a function such that  $D(\varphi) := \{x \in \mathbb{E} : \varphi(x) < +\infty\} \neq \emptyset$ . Give a definition of the *conjugate* function  $\varphi^* : \mathbb{E}^* \rightarrow (-\infty, +\infty]$ . As an example compute the conjugate function  $\varphi^* : \mathbb{R} \rightarrow (-\infty, +\infty]$  for  $\varphi : \mathbb{R} \rightarrow (-\infty, +\infty]$  given by

$$\varphi(x) = \begin{cases} -\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ +\infty & \text{if } |x| > 1. \end{cases}$$

# Complex Analysis Qualifying Exam

Spring 2016

April 11, 2016

1. Let  $f(z) = (3/2)x^2 - xy + ixy^2$ , where  $x = \operatorname{Re}(z)$ ,  $y = \operatorname{Im}(z)$ . Locate all points  $z \in \mathbb{C}$  at which  $f$  is complex-differentiable, and determine  $f'(z)$  for each such point. Is  $f$  analytic at these points?
2. Let  $f$  be a function analytic inside and on the unit circle. Suppose that  $|f(z) - z| < |z|$  on the unit circle.
  - (a) Show that  $|f'(1/2)| \leq 8$ .
  - (b) Show that  $f$  has precisely one zero inside the unit circle.
3. Find all entire functions  $f$  which satisfy  $\operatorname{Re}(f(z)) \leq \frac{2}{|z|}$  for  $|z| > 1$ . Prove your answer.
4. Use the calculus of residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9} dx.$$

Verify all steps of the calculation.

Abstract Algebra Qualifying Exam

April 8, 2016

Name \_\_\_\_\_

**Instructions.** Please solve any four problems from the list of the following problems (show all your work).

1. Let  $\varphi : G_1 \rightarrow G_2$  be a homomorphism of groups.
  - a) Prove that if  $\varphi$  is surjective and  $G_2$  is non-abelian then  $G_1$  is also non-abelian.
  - b) Prove that if  $\varphi$  is surjective and  $H_1$  is a normal subgroup of  $G_1$  then  $H_2 = \varphi(H_1)$  is a normal subgroup of  $G_2$ .
  - c) Prove that  $H = \{(g, \varphi(g)) \in G_1 \times G_2 \mid g \in G_1\}$  is a subgroup of  $G_1 \times G_2$ .
2. Prove that if  $|G| = pq$  for some primes  $p$  and  $q$  (not necessarily distinct) then either  $G$  is abelian or  $Z(G) = \{e\}$ .
3. Show that a group of order 56 cannot be simple.
4. Let  $G = D_6$  be the dihedral group of the rigid motions of a regular hexagon and  $X = \{H \subseteq G \mid H \leq G\}$ . Consider the action of  $G$  on  $X$  by conjugation, that is, for  $g \in G, H \in X$ :

$$g \cdot H = gHg^{-1}.$$

- a) Find the stabilizer  $G_H$  of  $H$ , for all  $H \in X$ .
  - b) Find the orbit  $G(H)$  of  $H$ , for all  $H \in X$ .
  - c) Find the set of all orbits  $X/G$ .
5. Prove that every finite integral domain is a field.
  6. Let  $I$  and  $J$  be ideals of  $R$ .
    - a) Prove that  $I + J$  is the smallest ideal of  $R$  containing both  $I$  and  $J$ .
    - b) Prove that  $IJ$  is an ideal contained in  $I \cap J$ .
    - c) Give an example where  $IJ \neq I \cap J$ .
    - d) Prove that if  $R$  is commutative ring with identity  $1_R$  and  $I + J = R$  then  $IJ = I \cap J$ .
  7. Let  $x^2 + x + 1$  be an element of the polynomial ring  $E = \mathbb{Z}_2[x]$ . Denote by  $\overline{E} = \mathbb{Z}_2[x] / (x^2 + x + 1)$  and  $\overline{a} = a + I$ , where  $I = (x^2 + x + 1)$ .
    - a) Show that  $\overline{E} = \{\overline{0}, \overline{1}, \overline{x}, \overline{x+1}\}$
    - b) Write out the  $4 \times 4$  addition table for  $\overline{E}$  and deduce that  $(\overline{E}, +)$  is isomorphic as an abelian group to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .
    - c) Write out the  $4 \times 4$  multiplication table for  $\overline{E}$  and deduce that  $(\overline{E} \setminus \{0\}, \cdot)$  is isomorphic as an abelian group to  $\mathbb{Z}_3$ .
    - d) Show that  $(\overline{E}, +, \cdot)$  is a field.

**Qualifying Exam: Ordinary Differential Equations I, April 2016**

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 34 points each Give clear and complete answers with full details in proofs

1. Consider the following IVP:

$$x'' - 2tx' - 3x = 0, \text{ with } x(0) = 1, x'(0) = 0.$$

- Write the above in the form of first order Ordinary Differential Equations.
- Write the found first order ODE in terms of an integral equation.
- Define a set of successive approximate solutions  $\{\phi_n\}$  for the the integral equation.
- Write out  $\phi_0, \phi_1$  explicitly, and simplify them as much as possible.
- Do you think the sequence  $\{\phi_n\}$  you have written converges or not? In either case, explicitly justify your answer.

2. Consider the boundary value problem on  $[0, \pi]$  for the equation

$$x'' + 4\lambda x = 0,$$

with  $x'(0) = 0$  and  $x(\pi) = 0$ .

- Are there any eigenvalues which are not real?
- Find the eigenvalues
- Find the corresponding eigenfunctions.

3. Find the solution to the system:

$$\begin{aligned} X_1' &= 3X_1 + 2X_2 + t \\ X_2' &= 2X_1 + 3X_2 - t \end{aligned}$$

with the initial condition that  $\mathbf{X}(0) = \xi$ .

# Examination Booklet

## QE Exam: Choice

6390: Dynamical Systems, 11/04/2016

Last Name: \_\_\_\_\_  
(print)

First Name: \_\_\_\_\_  
(print)

Signature: \_\_\_\_\_

Read all of the following information before starting the exam:

- There are **five** problems in this exam. You need to solve **three** problems for full grade.
- Show the significant steps of your work clearly.
- Circle or otherwise indicate your final answers.
- If you need to visit the restroom, bring your paper to the proctor.
- You may not leave the exam until 30 minutes have elapsed.
- Good luck!



**For Instructor's Use Only**

Question	Weight	Your score
1	33	
2	33	
3	33	
4	33	
5	33	
<b>Total</b>		

**Problem 1.** Show (rigorously) that the shift operator acting on binary sequences (the chopping map) features three properties of chaos:

1. A dense set of periodic orbits.
2. Sensitivity with respect to initial data.
3. An orbit which is dense everywhere.

You have to define a metric in the space of binary sequences for a rigorous proof.

**Problem 2.** Consider the predator-prey model with type II functional response, which is usually used to demonstrate Hopf bifurcation in ecological systems (the so-called “paradox of enrichment” in ecology first studied by Rosenzweig):

$$x' = x \left( 1 - \frac{x}{K} \right) - \frac{xy}{1+x}$$
$$y' = -\alpha y + \frac{xy}{1+x}$$

Assume that the predator death rate satisfies  $\alpha < 1$ .

1. Using the carrying capacity  $K$  as a parameter, find the Hopf bifurcation point  $K_H$  for the positive equilibrium of the system.
2. What is the approximate period of the cycle for  $K \approx K_H$ ?
3. Study stability of the positive equilibrium for  $K < K_H$  and  $K > K_H$ . Assuming that the Hopf bifurcation is supercritical, deduce the stability of the cycle from your analysis.

**Problem 3.** Consider a leaf-eating herbivores model

$$\begin{aligned}v_{n+1} &= kv_n e^{-ah_n} \\h_{n+1} &= rh_n \left( \delta - \frac{h_n}{v_n} \right)\end{aligned}$$

where  $h_n$  is the population of herbivores and  $v_n$  is the mass of leaves.

a. Show that by rescaling the equations it is possible to reduce the number of parameters. To do so, define

$$V_n = v_n/v_*, \quad H_n = h_n/h_*,$$

where  $(v_*, h_*)$  is the positive equilibrium of the system, and show that in these new variables the system becomes

$$\begin{aligned}V_{n+1} &= V_n e^{\alpha(1-H_n)} \\H_{n+1} &= \beta H_n \left( 1 + \frac{1}{\beta} - \frac{H_n}{V_n} \right)\end{aligned}$$

What is the relationship between the new parameters  $\alpha, \beta$  and the old parameters  $k, a, r, \delta$ ? Note that the positive equilibrium naturally becomes  $(V_*, H_*) = (1, 1)$ .

b. Determine the domain of stability of the positive equilibrium. Which bifurcations limit this domain?

**Problem 4.** A canonical model for host-parasitoid interaction is the Nicholson-Bailey model

$$\begin{aligned}N_{n+1} &= \mu N_n e^{-aP_n} \\ P_{n+1} &= cN_n(1 - e^{-aP_n})\end{aligned}$$

where  $N_n$  is the number (biomass) of hosts and  $P_n$  is the number (biomass) of parasitoids in generation  $n$ .

- a. Analyze stability of the zero equilibrium of this model.
- b. Find the positive equilibrium of this model and analyze its stability. Show that it is unstable. (This model generates increasingly large oscillations of hosts/parasites).

**Problem 5.** Consider the cubic equation

$$\varepsilon x^3 - x + 2 = 0$$

with a small  $\varepsilon > 0$ .

(i) Show that the straightforward expansion  $x = x_0 + \varepsilon x_1 + \dots$  gives only one of the three roots. Find the leading order and the first correction to this root.

(ii) Modify the perturbation expansion to obtain the leading order and the first correction to the second and third roots.

Qualifying Exam (CHOICE): Ordinary Differential Equations II, April 2016

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 34 points each Give clear and complete answers with full details in proofs

1. Let  $a(t) \neq 0$  be continuous,  $T$ -periodic function and let  $\phi_1$  and  $\phi_2$  be solutions of

$$y'' + a(t)y = 0$$

such that  $\phi_1(0) = \phi_2'(0) = 1$  and  $\phi_1'(0) = \phi_2(0) = 0$ . Define  $\alpha = -(\phi_1(T) + \phi_2'(T))$ . For what values of  $\alpha$  can you be sure that the trivial solution of above ODE is stable?

2. Consider the equation

$$y'' = -\sin y$$

Find all equilibrium points and analyze each for stability using appropriate Lyapunov functions.

3. Find the stable manifold and unstable manifold of  $X' = AX + F(X)$  where

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ and } F(X) = \begin{pmatrix} x_3^3 \\ 12x_3^5 \\ 0 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

and find their tangents at  $X = 0$ .

**MATH 6331 - QUALIFYING  
EXAMINATIONS  
Spring 2016**

- Do any 4 questions below. Each is worth 25 points.
1. Define a positive linear control system. Show that a linear system is positive iff  $b_{ij} \geq 0, \forall i, j$  and  $a_{ij} \geq 0, \forall i \neq j$ .
  2. Let  $f_0(x) = Ax$  and  $f_1(x) = b$ , where  $b$  is a constant vector in  $R^n$ . Compute the Lie brackets  $[f_0, f_1]$  and  $[f_1, [f_0, f_1]]$ . (7.5 points)
  3. State and prove Heymann's Lemma.
  4. State an integral representation for the unique positive definite solution to the matrix Lyapunov Equation  $A^T P + PA = C$ , where  $A$  is a given Hurwitz matrix and  $C$  is a negative definite matrix. You must also justify why the stated  $P$  is positive definite.
  5. State and derive the formula for i) the state space realization for the series connection of transfer functions and ii) the state space realization of the inverse of a transfer function.
  6. Consider the Algebraic Riccati equation  $A^T P + PA - PBR^{-1}B^T P + Q = 0$ , with  $R > 0, Q \geq 0$ . Define the Hamiltonian matrix associated to it and verify that it is indeed Hamiltonian. State what it means for it to be in the domain of the Riccati operator.



**MATH 6319 - Ph. D Qualifying Examinations**  
**Spring 2016**  
**V. Ramakrishna**

**CHOICE:** Do any 4 of the Qs below. Each Q is worth 25 points.

• **Q1**

- i) Let  $C \geq D$ , with  $C$  and  $D$  themselves positive semidefinite. Show that  $\det(C) \geq \det(D)$ .
- ii) Let  $X$  be a  $2 \times 2$  block matrix which is positive semidefinite. Assume the  $SE$  block,  $D$ , is invertible. Prove *Fischer's inequality*:  $\det(X) \leq \det(A)\det(D)$ , where  $A$  is the  $NW$  block of  $X$ .

• **Q2** Let  $X$  and  $Y$  be  $n \times n$  matrices. Prove that the characteristic polynomial of a  $2n \times 2n$  matrix similar to the matrix  $U = \begin{pmatrix} 0_n & X \\ Y & 0_n \end{pmatrix}$  is an even polynomial. i

• **Q 3**

Use the fundamental theorem of linear algebra (i.e., the rank-nullity theorem) to show that the solution to the Lagrange interpolation problem is unique and that it is indeed given as a linear combination of the Lagrange interpolating polynomials.

• **Q4**

- State the law of complementary nullities (all submatrices related notation must be carefully explained).
- Let  $X$  be  $5 \times 5$  invertible. Let the rank of  $X[\alpha, \beta]$ , where  $\alpha = \{1, 3, 5\}$ ,  $\beta = \{2, 3\}$  be 1. What does the law of complementary nullities say in this case about the rank of a certain submatrix of  $X^{-1}$ ? Your answer must state which submatrix also.

•**Q5** Show that if  $X$  is a Hamiltonian matrix then  $\exists S$ , symmetric, with  $X = J_{2n}S$ . (For additional 10 points: Can  $X$  be expressed as  $\tilde{S}J_{2n}$ , where  $\tilde{S}$  is another symmetric matrix? Justify your answer).

• **Q6** Let  $A$  and  $B$  two real matrices which are similar. Show that the similarity may be chosen to be real too.

**Q7** Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier transform.

MATH 6310 (TOPOLOGY) – QUALIFYING EXAM  
SPRING 2016

**Problem 1.** Given  $f, g, h$  in  $\pi_1(X, x_0)$ . Construct an explicit path homotopy between  $(f \cdot g) \cdot h$  and  $f \cdot (g \cdot h)$ .

**Problem 2.** a) Let  $A \subset X$ ; suppose  $r : X \rightarrow A$  is a continuous map such that  $r(a) = a$  for each  $a \in A$ . (The map  $r$  is called a *retraction* of  $X$  onto  $A$ .) If  $a_0 \in A$ , show that

$$r_* : \pi_1(X, a_0) \rightarrow \pi_1(A, a_0)$$

is surjective.

b) Show that there are no retractions of the Möbius band onto its boundary circle.

**Problem 3.** Compute the fundamental group of the space obtained from two tori  $S^1 \times S^1$  by identifying a circle  $S^1 \times \{x_0\}$  in one torus with the corresponding circle  $S^1 \times \{x_0\}$  in the other torus.

**Problem 4.** a) Use van Kampen's theorem to compute  $\pi_1(S^n)$  for  $n \geq 2$ .

b) Show that  $\mathbb{R}^2$  is not homeomorphic to  $\mathbb{R}^n$  for  $n \geq 3$ .

**Problem 5.** Show that if a path-connected, locally path-connected space  $X$  has  $\pi_1(X)$  finite, then every map  $X \rightarrow S^1$  is null-homotopic. [Hint: Use the covering map  $\mathbb{R} \rightarrow S^1$  and the lifting criterion.]