UTD University of Texas at Dallas

# Real Analysis

## Qualifying Exam, Spring 2015

Choose 5 questions and indicate your choice (by putting a check mark $\surd$) in the table below.

| Question | Weight | Your Score | Choice |
|---|---|---|---|
| 1. | 20 | | |
| 2. | 20 | | |
| 3. | 20 | | |
| 4. | 20 | | |
| 5. | 20 | | |
| 6. | 20 | | |
| 7. | 20 | | |
| 8. | 20 | | |
| Total: | **100** | | |

**Problem 1.** Denote by $B$ the closed unit ball in the Euclidean space $\mathbb{R}^n$. Put

$$\mathscr{A} := \{\varphi \in C(B,\mathbb{R}) : \varphi \text{ is continuously differentiable, } \max\{\|\nabla\varphi(x)\| : x \in B\} \leq 1,\ \varphi|_{\partial B} \equiv 0\}$$

Show that $\mathscr{A}$ is relatively compact in $C(B,\mathbb{R})$.

**Problem 2.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $g : [a, b] \to \mathbb{R}$ of bounded variation. Show that for every $x \in [a, b]$, the integral

$$F(x) := \int_a^x f(t)dg(t)$$

defines a function $F : [a, b] \to \mathbb{R}$ which is continuous at each point of continuity of the function $g$.

**Problem 3.** Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of *increasing* (i.e. non-decreasing) functions on $[a, b]$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Assume that $Z \subset [a, b]$ is a subset such that

(i) $a,\ b \in Z$,

(ii) $\overline{Z} = [a, b]$, i.e. $Z$ is dense in $[a, b]$,

(iii) For every $x \in Z$ we have $\lim_{n \to \infty} f_n(x) = f(x)$.

Prove that $f_n$ converges uniformly to $f$ on $[a, b]$.

**Problem 4.** Let $f : [0,1] \to \mathbb{R}$ be a function differentiable at any point. Is $f'$ a measurable function? Justify your answer.

## Problem 5.

(i) Show that

$$\int_{[0,1]}\left(\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}dy\right)dx=\frac{\pi}{4}\ \text{and}\ \int_{[0,1]}\left(\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}dx\right)dy=-\frac{\pi}{4}.$$

(ii) Does this contradict Fubini's theorem? Justify your answer.

**Problem 6.** Let $I := [0,1]$. Does there exist an open subset $B \subset I$ which is everywhere dense in $I$ and such that $I \setminus B$ has positive Lebesgue measure?

**Problem 7.** Consider the sequence

$$f_n(x) = \frac{n\sin(x)}{1 + n^2\sin^2(x)} \quad (x \in [0, \pi]).$$

(i) show that $\{f_n\}$ converges a.e.;

(ii) show that $\{f_n\}$ does not converge uniformly;

(iii) describe **explicitly** the set $A \subset [0, \pi]$ such that $\mu(A) < \frac{1}{2015}$ and $\{f_n\}$ converges uniformly on $[0, \pi] \setminus A$ (here "$\mu(\cdot)$" stands for the Lebesgue measure on $[0, \pi]$).

**Problem 8.** Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing function and put $D := f([a, b])$. Show that $f^{-1} : D \to \mathbb{R}$ is continuous.

UTD University of Texas at Dallas

# Functional Analysis I

## Qualifying Exam, Spring 2015

Choose 5 questions and indicate your choice (by putting a check mark $\surd$) in the table below.

| Question | Weight | Your Score | Choice |
|---|---|---|---|
| 1. | 20 | | |
| 2. | 20 | | |
| 3. | 20 | | |
| 4. | 20 | | |
| 5. | 20 | | |
| 6. | 20 | | |
| 7. | 20 | | |
| 8. | 20 | | |
| Total: | **100** | | |

**Problem 1.** Let $\mathbb{E}$ be a normed space and $L \subset \mathbb{E}$ be a linear subspace. We put

$$L^{\perp} := \{f \in \mathbb{E}^* : \forall_{x \in L} \ \langle f, x \rangle = 0\}.$$

Show that

$$\operatorname{dist}(f, L^{\perp}) := \sup\{\langle f, x \rangle : \|x\| \leq 1 \ \text{ and } \ x \in L\}.$$

**Problem 2.** Let $\mathbb{E}$ be a Banach space and $f \in \mathbb{E}^*$. Show that

$$\operatorname{dist}(x, \operatorname{Ker}(f)) = \frac{|f(x)|}{\|f\|}.$$

**Problem 3.** Let $\mathscr{H}$ be a Hilbert space.

(a) Show that for two vectors $x$, $y \in \mathscr{H}$ we have the following:

$$x \perp y \quad \Leftrightarrow \quad \forall_{t \in \mathbb{R}} \ \|x + ty\| \geq \|x\|.$$

(b) Assume that $P \in L(\mathscr{H}, \mathscr{H})$ is a projection, i.e. $P^2 = P$. Show that $P$

$$\forall_{x \in \mathscr{H}} \ Px \perp x - Px \ \Leftrightarrow \ \|P\| = 1,$$

i.e. $P$ is an orthogonal projection if and only if $\|P\| = 1$.

**Problem 4.** Let $\mathbb{E}$ be a normed space and $L \subset \mathbb{E}$ a subspace of co-dimension 1.

(a) Show that there exists $f \in \mathbb{E}^*$ such that $\text{Ker}(f) = L$.

(b) Find a projection $P \in L(\mathbb{E}, \mathbb{E})$ onto $L$, i.e. $P^2 = P$ and $\text{Im}(P) = L$.

**Problem 5.** Does there exist a Banach space $\mathbb{E}$ such that the space $\mathbb{E}^*$ (its conjugate) admits an infinite *countable* basis? Justify your answer.

**Problem 6.** Let $M \subset l_2$ be the set of all sequences $x = (x_1, ..., x_n, ...)$ satisfying the condition:

$$\sum_{n=1}^{\infty} n^2 x_n^2 < 1.$$

Show that $M$ is a convex set, but not a convex body. (Recall, in a normed space a convex set is a *convex body* if it admits at least one interior point).

**Problem 7.** Let $B$ be the space of all continuous functions on the segment $[0,1]$ equipped with the norm

$$\|f\| := \left( \int_0^1 |f(x)|^4 dx \right)^{\frac{1}{4}}.$$

(i) Is $B$ separable?

(ii) Is $B$ complete?

(iii) Does there exist a normed space $\mathbb{E}$ such that $\mathbb{E}^* = B$?

Justify your answers.

**Problem 8.** Does any bounded weakly convergent sequence in $l^2$ converge coordinate-wisely? Justify your answer.

Name: ______________________________

# Complex Analysis Qualifying Exam
# Spring 2015
# Vladimir Dragović and Tobias Hagge

April 17, 2015

**Definition 1** *The cross ratio* of four complex numbers is defined as:

$$(z_1, z_2, z_3, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

**Definition 2** A *homeomorphism* between two metric spaces is a continuous, injective, and surjective map with the inverse which is a continuous map.

1. Find if it exists a conformal bijection between the upper half plane and each of the following regions:
   (a) The Unit disk $D = \{x \in \mathbb{C} | \, |x| < 1\}$,
   (b) $D - D_2$, where $D_2 = \{x \in \mathbb{C} | \, |x - \frac{1}{2}| \leq \frac{1}{2}\}$.

2. (a) Call all lines and circles of the plane *the extended circles*. Prove that a fractional linear transformation maps extended circles to extended circles.
   (b) If $f$ is a fractional linear transformation, prove that it preserves the cross-ratio: $(f(z_1), f(z_2), f(z_3), f(z_4)) = (z_1, z_2, z_3, z_4)$.

3. Let $f$ be an entire function.
   (a) Show that if $f$ has a pole or a removable singularity at infinity, $f$ is a polynomial.
   (b) Show that if $f$ is a homeomorphism, then $f(z) = az + b$ for some $a \in \mathbb{C}^{\times}$, $b \in \mathbb{C}$.

4. If $D$ is the open unit disk centered at the origin, $\Omega \subset \mathbb{C}$ a connected open set, show that for any two holomorphic homeomorphisms $f : D \to \Omega$ and $g : D \to \Omega$ such that $f(0) = g(0)$, there exists a constant $k$ such that for all $z \in D$, $f(z) = g(kz)$. What can you say about $k$?

5. Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial with complex coefficients, where each $|a_i| = 1$. Show that all zeros of $p$ are contained in the closed disk $D_2$ of radius 2 and centered at the origin.

6. Calculate the integral, carefully justifying each step:

$$\int_0^{\infty} \frac{dx}{1 + x^7}.$$

   (Hint: One may apply the residue calculus to the integrals along the curves $\gamma_R = I_R + C_R - I_{w^2R}$, where $w$ is the root of the equation $z^7 = -1$, with the smallest positive argument; $I_R$ is the segment from the origin to the point $(R, 0)$; $I_{w^2R}$ is the segment from the origin to the point $w^2R$. $C_R$ is a part of the circle of radius $R$ centered at the origin, connecting the points $(R, 0)$ and $w^2R$.)

**Name:** ______________________________

# Abstract Algebra Qualifying Exam
## Spring 2015
## Tobias Hagge

April 15,2015

Show all work.

1. Classify all abelian groups of order 60 up to isomorphism.
2. Let $|G|$ be a group of order 30.
   (a) Show that $G$ is not simple.
   (b) Show that $G$ has a cyclic subgroup of order 15.
3. Let $G$ be a finite group, $H \leq G$ such that $[G : H]$ is the least prime which divides $|G|$. Show that $H \trianglelefteq G$.
   Suggested outline: let $X$ the set of subgroups of $G$ which are conjugate to $H$ in $G$. Argue that if $|X| > 1$, $G$ acts trivially on $X$, and obtain a contradiction.
4. (a) Show that if $R$ is a field, the polynomial ring $R[x]$ is a Euclidean domain.
   (b) Show that $\mathbb{Z}[x]$ is not a Euclidean domain.
5. Show that in a principal ideal domain, a prime ideal is maximal.
6. Let $R$ be a ring with 1, $M$ an $R$-module, $N$ a submodule of $M$. Prove that if $N$ and $M/N$ are finitely generated then so is $M$.

**Qualifying Exam: Ordinary Differential Equations I, April 2015**
**Math 6315**

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Give clear, rigorous and complete answers with full details in proofs

1. Consider the system

$$\begin{aligned} x_1' &= a(t)x_2 \\ x_2' &= b(t)x_1 \end{aligned}$$

for $t \geq 0$. Assume $a$ & $b$: $\mathbf{R}_+ \to \mathbf{R}$ are continuous functions with $\lim_{t\to+\infty} a(t) = 1$ and $b(t)$ is absolutely integrable on $\mathbf{R}_+$. Prove that:
(**a.** ) A unique solution to the above system of ODE with initial values $x_1(\tau) = \xi_1$ and $x_2(\tau) = \xi_2$ exist for all positive and finite values of $t$.
(**b.** ) If a solution to above system of ODE is such that $x_1(t)$ is bounded on $\mathbf{R}_+$, then

$$\lim_{t\to+\infty} x_2(t) = 0$$

(**c.** ) Make use of the above, to show that the given system of ODE has at least one solution which is unbounded on $\mathbf{R}_+$.

2. Find the solution to the following initial value problem, by first finding its state transition matrix.

$$\begin{aligned} x_1' + 2x_1 + x_2 &= \sin t \\ x_2' - 4x_1 - 2x_2 &= \cos t \end{aligned}$$

3. Consider the BVP

$$y''' + \lambda y = 0, \quad y(0) = y'(0) = y''(1) = 0, \quad 0 \leq t \leq 1$$

where $y' = dy/dt$.
(**a.**) Is $\lambda = 0$ an eigenvalue of this BVP?
(**c.**) Compute the Green's function, when $\lambda = 0$.
(**c.**) Is this problem self-adjoint?

# Numerical Analysis Qualifying Exam

Do any 4 out of 6 problems

1) Suppose that $f$ is a function on $[0,3]$ for which one knows that

$$f(0)=1\ ,\ f(1)=2\ ,\ f^{'}(1)=-1\ ,\ f(3)=f^{'}(3)=0.$$

Estimate $f(2)$, using Hermite interpolation.

2) Determine the quadrature formula of the type

$$\int_{-1}^{1} f(t)dt = a_{-1}\int_{-1}^{-\frac{1}{2}} f(t)dt + a_0 f(0) + a_1 \int_{\frac{1}{2}}^{1} f(t)dt + E(f)$$

having maximum degree of exactness $d$. What is the value of $d$?

3) Consider the fixed point iteration

$$x_{n+1}=\phi(x_n)\ ,\ n=0,1,2,\ldots\ ,\ \text{where } \phi(x)=Ax+Bx^2+Cx^3.$$

Given a positive number $\alpha$, determine the constants $A, B, C$ such that the iteration converges locally to $\frac{1}{\alpha}$ with order $p=3$.

4) Let $g(x,y)=(f_x+f_y f)(x,y)$. Show that the one-step method defined by the increment function

$$\Phi(x,y;h)=f(x,y)+\frac{1}{2}hg(x+\frac{1}{3}h, y+\frac{1}{3}hf(x,y))$$

has order $p=3$. Express the principal error function in terms of $g$ and its derivatives.

5) Describe how Newton's method is applied to solve the system of nonlinear equations

$$u_{n+k}=h\beta_k f(x_{n+k},u_{n+k})+g_n\ ,\ g_n=h\sum_{s=0}^{k-1}\beta_s f_{n+s}-\sum_{s=0}^{k-1}\alpha_s u_{n+s}$$

for the next approximation, $u_{n+k}$.

6) Show that the boundary value problem

$$y^{''}+e^{-y}=0\ ,\ 0\le x\le 1\ ,\ y(0)=y(1)=0$$

has a unique solution that is nonnegative on $[0,1]$. Set up a finite difference method for solving the problem numerically. Discuss the convergence of your scheme.

Qualifying Exam: Choice- Ordinary Differential Equations II
April 2015, Math 6316

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Give clear, rigorous and complete answers with full details in proofs

1. (a) Determine all the equilibrium points of

$$\begin{aligned} x_1' &= x_2 + x_1 x_2 \\ x_2' &= -x_1 + 2x_2. \end{aligned}$$

(b) Determine the stability of $\mathbf{x} = 0$ for the nonlinear system given in part(a) by finding an appropriate Lyapunov function.

2. (a) Analyze the stability properties of the equilibrium point $x = 0$ for the following equation.

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} (e^{x_1} - 1)\sin(x_2 t) \\ e^{-t} x_1 x_2 \end{pmatrix}$$

(b) For the above nonlinear system, find the tangent spaces to its stable and unstable manifolds at the origin (if they exist).

3. Does the following 2 dimensional system

$$x_1' = -x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$x_2' = x_1 + x_2(1 - x_1^2 - x_2^2)$$

have a periodic solution? If so, is it orbitally stable? Hint: It may be helpful to look at the problem in polar coordinates.

# Topology and Geometry (6306), Qualifying Exam

Friday, April 17th, 2015

1. Given a continuous function $f : X \to Y$, $X$ is compact.

   a Prove that $f(X)$ is compact.

   b If $Y$ is Hausdorff and $f$ is a bijection, prove that $f$ is homeomorphism.

2. a Let $f, g : X \to Y$ be two continuous maps and assume that $Y$ is Hausdorff. Prove that the set $\{x|f(x) = g(x)\}$ is closed in $X$.

   b Prove that $Y$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x)|x \in Y\} \subset Y \times Y$ is closed.

3. Prove that the circle $S^1 = \{(x, y)|x^2 + y^2 = 1\}$ is a smooth compact manifold. Does it have a structure of Lie group? Is it connected?

4. Calculate the induced metric (the first fundamental form) on the surface in $R^3$

   $$r(u, v) = (a \cos u \cos v, a \sin u \cos v, a \sin v).$$

   What kind of surface it is? What is the Gauss curvature at the point $u = \pi/3, v = \pi/4$?

5. Given a metric space $(X, d)$. A map $f : X \to X$ is isometry if for all $x, y \in X$ the condition is satisfied

   $$d(f(x), f(y)) = d(x, y).$$

   If $X$ is compact prove that every isometry $f$ is bijective and a homeomorphism. Show that if $X$ is not compact, than an isometry is not always a homeomorphism.

# Differential Geometry (6309), Qualifying Exam

Friday, April 17th, 2015

1. Calculate the induced metric (the first fundamental form) on the surface in $R^3$
$$r(u,v) = (a\cos u \cos v, a\sin u \cos v, a\sin v).$$
What kind of surface it is? What is the Gauss curvature at the point $u = \pi/3, v = \pi/4$?

2. Compute the Christoffel symbols of the surface of revolution
$$\mathrm{r}(u,v) = (g(u)\cos v, g(u)\sin v, h(u)), \quad (g'(u))^2 + (h'(u))^2 = 1.$$

3. Prove that the circle $S^1 = \{(x,y)|x^2+y^2=1\}$ is a smooth compact manifold. Does it have a structure of Lie group? Is it connected?

4. Find the critical points and critical values of the maps:
   (a)
   $$f : \mathbb{R}^3 \to \mathbb{R}^2, (x,y,z) \mapsto (x+y^2, y+z^2);$$
   (b)
   $$f : \mathbb{R}^3 \to \mathbb{R}^2, (x,y,z) \mapsto (xy, z).$$
   (c) Given two vector fields $v_1 = x\partial_x + y\partial_y$ and $v_2 = (x-y)\partial_x + x\partial_y$. Calculate the commutator of $v_1$ and $v_2$.

5. Given a metric space $(X,d)$. A map $f : X \to X$ is isometry if for all $x, y \in X$ the condition is satisfied
$$d(f(x), f(y)) = d(x,y).$$
If $X$ is compact prove that every isometry $f$ is bijective and a homeomorphism. Show that if $X$ is not compact, than an isometry is not always a homeomorphism.

# MATH 6320 - QUALIFYIING "CHOICE" EXAMINATION
# For: Ms. Het Mankad
# April 17th 2015
# Spring 2015
# V. Ramakrishna

- You must show all work to receive full credit. Each Q is worth 20 points.

- No calculators or electronic devices (including phones) allowed.

- Use only **pens**. No pencils allowed.

- Write your answers only in the space provided, i.e., not on margins, not in the space separating two parts of a question. Further, **CLEARLY** separate answers to each subpart. Material written at points where they do not belong, or as an afterthought, will be **ignored**.

- **Q 1** Show that $A_{MPI}$ is the matrix $X$ which minimizes $\|\ I - AX\ \|_F$.

**Q2** State four conditions equivalent for a matrix to be a RPN matrix.

**Q3** Suppose $A_{m \times n}$ has rank $r$. Suppose further that the first $r$ left singular vectors, $u_i$, of $A$ have been found. State and justify a formula for the first $r$ right singular vectors, $v_i$, of $A$, which does not require solving an eigenvalue problem.

**Q4** Let $A$ be a square matrix of rank $r$. Show that there is a polynomial of degree $r+1$ which annihilates $A$.

**Q5** Let $C$ be the commutator of two square matrices $A$ and $B$. Show that if $C$ commutes with either matrix then $C$ is nilpotent. Use this to show that a matrix $A$ is normal iff it commutes with $C = [A, A^*]$.