REAL ANALYSIS

QUALIFYING EXAM, SPRING 2014; CLOSED BOOKS

Solve four of the following five problems

1. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function.

(i) Is it true that f' is necessarily continuous? Justify your answer.

(ii) Is it true that f' is necessarily measurable? Justify your answer.

2 Compute $\int_{[0,\pi/2]} f d\mu$, where

$$f(x) = \begin{cases} \cos(x), & \text{if } \sin(\mathrm{x}) \text{ is rational;} \\ \cos^3(x), & \text{if } \sin(\mathrm{x}) \text{ is irrational} \end{cases}$$

3. Let $f : [a,b] \to \mathbb{R}$ be a measurable function such that $\displaystyle\int_a^b |f| d\mu = 0$. Show that $f(x) = 0$ a.e.

4. Let $f : [a,b] \to \mathbb{R}$ be a summable function and $\{c_n\}$ an increasing sequence such that $a < c_n < b$ for all $n \in \mathbb{N}$. Put $c := \lim_{n\to\infty} c_n$. Show that

$$\int_{c_1}^{c} f d\mu = \sum_{n=1}^{\infty} \int_{c_n}^{c_{n+1}} f d\mu.$$

5. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function (i.e. $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$). Assume, in addition, that f is continuous. Show that f is linear.

(ii) Is Statement (i) true if we replace continuity by measurability? Justify your answer.

REAL FUNCTIONAL ANALYSIS (MATH 6302)

QULIFYING EXAM, SPRING 2014; CLOSED BOOKS

Solve four of the following five problems

1. Let E be a Banach space. Show that E is finite-dimensional if and only if every closed bounded subset of E is compact.

2. Let E be a normed space, $A, B \subset E$ and $A + B := \{a + b \ : \ a \in A \ , \ b \in B\}$.

(i) Assume A is open and B is arbitrary. Show that $A + B$ is open.

(ii) Assume A is closed and B is compact. Show that $A+B$ is closed.

(iii) Assume A is closed and B is closed. Is $A+B$ necessarily closed? Justify your answer.

3. Let H be the space of real continuous functions on $[-1, 1]$ equipped with the inner product $< f, g >:= \int_{-1}^{1} f(t)g(t)dt$. Let $K \subset H$ be the linear subspace of H consisting of functions which are equal to zero at zero. Find the orthogonal complement of K in H (i.e., the linear space of functions from H orthogonal to any function from K). Justify your answer.

4. Let $A : E \to F$ be a linear operator between two normed spaces. Show that A is bounded if and only if there exists an open non-empty set $U \subset E$ such that $A(U)$ is bounded in F.

5. Let c_o be the space of real sequences converging to zero and equipped with the sup-norm. Let c_o^* stand for the conjugate (dual) space of linear continuous functionals on c_o equipped with the standard norm: if $f \in c_o^*$, then:

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

Show that c_o^* is isometrically isomorphic to the space l^1 of real sequences $\{x_n\}$ with $\sum_n |x_n| < \infty$ equipped with the norm $\sum_n |x_n|$.

Qualifying Exam: Ordinary Differential Equations I, April 2013
Math 6315

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Give clear and complete answers with full details in proofs

1. Let $A(t) \in C(-\infty, \infty)$ be periodic with period T, i.e. $A(t+T) = A(t)$.
[**a**] (25 points) Prove that the fundamental set of solutions $\Phi(t)$ for the ODE $\mathbf{x}' = A(t)\mathbf{x}$ for $-\infty < t < \infty$ can be represented as $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is a periodic nonsingular matrix with period T and R is a constant matrix.
[**b**] (25 points) Find the fundamental set of solutions for $x'' = \sin(t)x'$. Find R and explicitly show that the associated 2×2 matrix $P(t)$ for this problem is periodic.

2. (25 points) Suppose for a given continuous function $f(t)$ we find the equation
$$\mathbf{x}' = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix} \mathbf{x} + f(t)$$
has a solution $\phi(t)$ which satisfies: $sup\{|\phi(t)| : \tau \leq t < \infty\} < \infty$. Prove that all other solutions to above ODE must satisfy the same boundedness condition.

3. (25 points) Consider the boundary value problem on $[0, 2]$ for the ODE
$$Lx = \lambda x, \text{ where } Lx = ix', \text{ and } x(2) = \alpha x(0)$$
with α being a constant complex number and $i = \sqrt{-1}$. Find the associated eigenvalues and eigenfunctions of this problem. For what values of α the eigenvectors are orthogonal. Please clearly justify your answer.

Qualifying Exam: Ordinary Differential Equations II, April 2014
Math 6316

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Give clear and complete answers with full details in proofs

1. [**a**] (25 points) Determine all the equilibrium point of

$$\begin{aligned} x_1' &= -x_2 + 2\sin x_1 \\ x_2' &= x_1 - x_1 x_2^2. \end{aligned}$$

   [**b**] (25 points) Determine the stability of $x = 0$ for the nonlinear system given in part(a) by finding an appropriate Lyapunov function.

2. [**a**] (25 points) Analyze the stability properties of the equilibrium point $x = 0$ for the following equation.

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} (e^{x_1} - 1)\sin(x_2 t) \\ e^{-t} x_1 x_2 \end{pmatrix}$$

   [**b**] (25 points) For the above nonlinear system, find the tangent spaces to its stable and unstable manifolds at the origin (if they exist).

Name: ____________________

Abstract Algebra Qualifying Exam
Spring 2014
Tobias Hagge

April 9,2014

Do as many as you are able. Show all work.

1. Let G be a group. Show that if $g^2 = e$ for all $g \in G$, then G is abelian.
2. Let G be a group. Show that if $G/Z(G)$ is cyclic, then G is abelian.
3. Show that if $(n_1, \ldots, n_k)$ is a sequence of relatively prime positive integers, then $\mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_k\mathbb{Z}$ is a cyclic group.
4. Let G be a group of order 12. Show that either G has a normal subgroup of order 3, or $G \cong A_4$ (hint: consider action by conjugation on the set of Sylow 3-subgroups).
5. Let T be a the planar region bound by an equilateral triangle with barycenter at the origin. The symmetric group S_3 acts by reflections and rotations on T, such that the action by permutation σ on T reduces to permutation by σ of the vertices. Up to isomorphism, is every transitive S_3-set present as an orbit of this action?
6. Show that no group of order 14 is simple.
7. Show that in a finite commutative ring R with $1 \neq 0$, every prime ideal is maximal.
8. Let R be a commutative ring with $1 \neq 0$. Suppose that every nonzero proper ideal of R is maximal. Show that if R has two distinct maximal ideals, then there are fields F_1 and F_2 such that $R \cong F_1 \oplus F_2$.
9. Show that if R is a ring and M is an irreducible R-module, then the ring $End_M(R)$ is a division ring.
10. Show that every module over a field is free.

Name: ______________________________

Complex Analysis Qualifying Exam
Spring 2014
Tobias Hagge

April 11, 2014

1. Let $a, b \in \mathbb{C}$, $k > 0$. Show that the set of points $\{z \in \mathbb{C} | |z - a| = k|z - b|\}$ is a line or circle.
2. Show that any meromorphic function on the plane with a finite limit at infinity is a rational function.
3. Prove the fundamental theorem of algebra using the argument principle. In particular, show that if f is a polynomial of degree $n > 0$ and D is a sufficiently large disk centered at the origin, f contains n zeros with multiplicity.
4. Show that every conformal homeomorphism of the upper half plane can be expressed in the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$.

5. Compute the residue of the function $\cot z$ at $z = 0$.
6. Compute $\int_0^\infty \frac{\cos x}{x^2+1} dx$ using complex methods.

Numerical Linear Algebra Qualifying Exam 4/11/2014
Dr. Minkoff

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

(1a) If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

what are the singular values of A?

(b) If the matrix of left singular vectors is

$$V = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

what is the minimum length least squares solution x^+ to $Ax = b$ if

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}?$$

(c) Prove the theoretical formula you used in Part (b) gives the minimum norm least squares solution.

(2) Assume that $A \in \mathbb{R}^{n\times n}$ has n linearly independent eigenvectors $v_1, \ldots v_n$. Let $\lambda_1, \ldots \lambda_n$ denote the eigenvalues associated with the eigenvectors $v_1, \ldots v_n$. Show that the sequence $(q_j) = (A^j q/\lambda_1^j)$ generated by the power method converges to the dominant eigenvector v_1 with convergence ratio $r = |\lambda_2/\lambda_1|$, provided that $|\lambda_1| > |\lambda_2| > |\lambda_3|$, and $c_1 \neq 0$ and $c_2 \neq 0$.

(3) Let

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}.$$

Find the QR factorization of A using
(a) Householder transforms,
(b) Givens rotations.

(c) Which of these two algorithms would be more efficient to use for this factorization if your matrix were large?

(d) Assuming $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, use one of your QR factorizations above to find the solution x to the system $Ax = b$.

(4a) Show that for any induced matrix norm, $\kappa(A) \geq 1$.

(b) If $Ax = b$ and $(A + \delta A)(x + \delta x) = b$, prove the inequality

$$\frac{||\delta x||}{||x + \delta x||} \leq \kappa(A) \frac{||\delta A||}{||A||}$$

where $\kappa(A)$ is the condition number of the matrix A.

(c) Verify the inequality for the system

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

using

$$\Delta A = \begin{pmatrix} 0 & 0 & 0.00003 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(d) Is the determinant of a matrix a good measure of the condition of a matrix? Give an example to help justify your answer.