REAL ANALYSIS MATH 6301
(Dr. Zalman Balanov)
Qualifying Exam
April, 2013

Name____________________

Instructions. *Please solve any* **five problems** *from the list of the following problems (show all your work).*

1. Let $I = [a,\ b] \subseteq \mathbb{R}$ be an interval and μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that for all $A \subseteq I$ and $B \subseteq I$, we have
$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B),$$
where $A \triangle B = (A \backslash B) \cup (B \backslash A)$ denotes the symmetric difference of sets. (Hint: Apply subadditivity property of an outer measure.)

2. Let $f_n : [a,\ b] \to \mathbb{R}$, $n = 1, 2, 3, \ldots$ be a sequence of measurable functions. Suppose there is a function $f : [a,\ b] \to \mathbb{R}$ such that
$$f(x) = \inf \{f_n(x) \mid n \in \mathbb{N}\}$$
Show that $f : [a,\ b] \to \mathbb{R}$ is measurable.

3. Let $E_1 \subseteq E_2 \subseteq \ldots \subseteq I$, where $I = [a,\ b] \subseteq \mathbb{R}$ is an interval, be a sequence of Lebesgue measurable subsets of I. Let $f : I \to \mathbb{R}$ be a Lebesgue integrable function. Show that
$$\lim_{n\to\infty} \int_{E_n} f\, d\mu = \int_E f\, d\mu, \text{ where } E = \bigcup_{n=1}^{\infty} E_n.$$

4. Suppose (X, μ) is a measure space with $\mu(X) = 1$ and let f and g be positive measurable functions on X such that $fg \geq 1$. Prove that
$$\left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right) \geq 1.$$

5. Does there exist an open everywhere dense subset $A \subset [0, 1]$ such that the complement of A in $[0, 1]$ has positive Lebesgue measure? Justify your answer.

6. Let $I = [a,\ b] \subseteq \mathbb{R}$ be an interval and let μ denote Lebesgue measure. Let $h : [a,\ b] \to [0,\ \infty]$ be measurable. If $A = \int_{[a,\ b]} h\, d\mu$, prove that
$$\sqrt{1 + A^2} \leq \int_I \sqrt{1 + h^2}\, d\mu \leq 1 + A.$$

REAL FUNCTIONAL ANALYSIS MATH G 302
(Dr. Zalman Balanov)
Qualifying Exam
April, 2013

Name________________________

Instructions. *Please solve any* **five problems** *from the list of the following problems (show all your work).*

1. Let E be a normed space. Prove that a linear functional $f : E \to \mathbb{R}$ is continuous if and only if $\ker f := \{x \in E \,:\, f(x) = 0\}$ is closed.

2. Give an example of a Banach space X containing two closed subspaces $U, V \leq X$ such that

$$U + V := \{u + v \,:\, u \in U \,\&\, v \in V\}$$

is not closed.

3. Show that every weakly convergent sequence in a normed linear space is bounded.

4. Let E be an infinite dimensional Banach space and let $L : E \to E$ be a completely continuous linear operator (i.e., it maps any bounded set into a relatively compact set). Is it possible that L is invertible? Justify your answer.

5. Let C be a closed convex (nonempty) set in a Hilbert space H. Show that for every $x_o \in H$ there exists a unique $y_o \in C$ such that $\|x_o - y_o\| = \text{dist}(x_o, C) := \inf\{\|x_o - y\| \,:\, y \in H\}$.

6. Show that any Hilbert space admits an orthonormal basis.

   Reminder: In a Hilbert space H, an orthonormal basis B is a family $\{e_\alpha\}$ of elements of H satisfying the conditions:

   (i) *Orthogonality:* Every two different elements of B are orthogonal;

   (ii) *Normalization:* Every element of the family has norm 1;

   (iii) *Completeness:* The linear span of the family B is dense in H.

Qualifying Exam: Ordinary Differential Equations I, April 2013

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 34 points each. Give clear and complete answers with full details in proofs

1. For $x \in \mathbf{R}$, consider the IVP $x' = -x + g(t)$, $x(0) = \xi$, where g(t) is continuous and $|g(t)| \leq 1$ for all t. Show that the IVP has a unique and bounded solution for all $t \geq 0$.

2. Consider the boundary value problem on $[0, \pi]$ for the equation

$$x'' + \lambda x = 0,$$

with $x'(0) = 0$ and $x(\pi) = 0$.

   a. Are there any eigenvalues which are not real?
   b. Find the eigenvalues
   c. Find the corresponding eigenfunctions.

3. a) Find e^{At}, for

$$A = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}.$$

b) Solve the following system of linear differential equations.

$$x_1' + 2x_1 + x_2 = \sin t$$
$$x_1' - 4x_1 - 2x_2 = \cos t$$

when $x_1(0) = 1$ and $x_2(0) = 0$

Qualifying Exam: Ordinary Differential Equations II, April 2013

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 34 points each Give clear and complete answers with full details in proofs

1. Find the stable manifold and unstable manifold of $X = AX$ where

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}. \; and \; X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2. Consider the equation

$$y'' = -\sin y$$

   Find all equilibrium points and analyze each for stability using appropriate Lyapunov functions.

3. Find the periodic orbits and determine their orbital stability for the system

$$r' = rf(r^2), \qquad \theta' = 1$$

   when $f(s) = |sin(s)|$.

Name: ______________________________

Abstract Algebra Qualifying Exam
Spring 2013
Tobias Hagge

April 10,2013

Choose six. Show all work.

1. Let G be a finite group of order n. Prove that if G is abelian but not cyclic, there is a proper divisor p of n such that $|g|$ divides p for every $g \in G$.
2. Prove that no group of order 48 is simple.
3. Let G be a finite group of order 35, acting on a finite set X of order 15. Compute all possible values for the cardinality of the fixed set X_G.
4. Let G be a finite group, $H_1 \leq G$, $H_2 \leq G$. Show that with the natural left action by multiplication, $G/H_1 \cong G/H_2$ as G-sets iff H_1 and H_2 are conjugate.
5. Let R be a commutative ring with $1 \neq 0$. Prove that every maximal ideal $I \trianglelefteq R$ is prime.
6. Prove that every finite integral domain is a field.
7. Let R be an integral domain such that every R-module is free. Prove that R is a field (Hint: consider the field of fractions).

Name: ______________________________

Complex Analysis Qualifying Exam
Spring 2013
Tobias Hagge

April 12, 2013

1. Give the most general form of a fractional linear transformation which sends the unit circle to itself.
2. Show that an entire function which is not a polynomial has an essential singularity at infinity.
3. Show that a power series expansion for a rational function has radius of convergence equal to the distance to the nearest pole.
4. Show that every conformal homeomorphism of the unit disk is a fractional linear transformation.
5. Let D be the unit disk. Compute $\int_D \frac{e^z}{\sin^3 z} dz$ by complex methods.
6. Compute $\int_{-\infty}^{\infty} \frac{\sin x}{x^3+x} dx$ by complex methods.

Topology
(Dr. Mieczyslaw K. Dabkowski)
Qualifying Exam
April 12, 2013

Name____________________

Instructions. *Please solve any* **four problems** *from the list of the following problems* (*show all your work*):

1. Prove that if $d : X \times X \to \mathbb{R}$ is a metric, then

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a metric.

2. Solve one of the following problems

a) Show that every closed subset of a compact topological space (X, τ) is compact.

b) Show that a compact subset of a T_2 (i.e. Hausdorff) topological space (X, τ) is closed.

c) Show that a subset of R^n is compact if and only if it is closed and bounded.

3. Let τ be the topology on $\mathbb{R}$ generated by the basis $\mathcal{B} = \{[-x, x] \mid x > 0\}$.

a) Determine: $\overline{\{0\}}$, $Int(\{0\})$, $\{0\}'$, and $\overline{\{1\}}$, $Int(\{1\})$, $\{1\}'$ in $(\mathbb{R}, \mathcal{T})$, where $\overline{X}$, $Int(X)$, X', denote the closure of X, nterior of X and set of the limit points of X respectively. No work necessary.

b) Is $\mathbb{R}$ connected under this topology? Is $\mathbb{R}$ compact under this topology? Explain why.

c) Can you compare τ with the standard topology on $\mathbb{R}$? That is, is one finer than the other?

d) Is $(\mathbb{R}, \tau)$ metrizable? (we may use hint: is it Hausdorff?)

e) Can the two intervals $[0, 1)$ and $(0, 1]$ be homeomorphic if they are given the subspace topology induced from τ? (hint: are they both compact?)

4. Show that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in $X \times X$.

5. Let A be a countable subset of $\mathbb{R}^n$, where $n > 1$. Show that $\mathbb{R}^n \backslash A$ is path-connected.

6. Let $\{A_\alpha \mid \alpha \in \Gamma\}$ be collection of nonempty connected subsets of a space X such that $X = \cup A_\alpha$.

a) Assume in addition that each A_α is open. Prove: X is connected if and only if for any α and β in Γ, there exist $\alpha = \alpha_1, \cdots, \alpha_n = \beta$ in Γ such that $A_{\alpha_i} \cap A_{\alpha_{i+1}} \neq \emptyset$.

b) If we drop the openness condition in (a), then only half of the "if and only" theorem in (a) holds. Which one is it? Show by example that the other direction does not hold.

MATH 6319 - Ph. D Qualifying Examinations
For:
April 12th, 2013
Spring 2013
V. Ramakrishna

- **Q 1**

Use the fundamental theorem of linear algebra to show that the solution to the Lagrange interpolation problem is unique and that it is indeed given as a linear combination of the Lagrange interpolating polynomials.

(10 points)

Q2 Show that if X is a Hamiltonian matrix then $\exists S$, symmetric, with $X = J_{2n}S$. Can X be expressed as $\tilde{S}J_{2n}$, where $\tilde{S}$ is another symmetric matrix?

(6 points)

Q3 Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier tarnsform.

(6 points)

Q4 Define an orthosymmetric bilinear/sesquilinear form. State three other conditions equivalent to it. Prove any one of these **four** equivalences of your choice.

(9 points)

MATH 6320 - Ph. d Qualifying Examinations For:

April 12th 2013
Spring 2013
V. Ramakrishna

Q1 Let $A \in M(n, \mathbf{C})$. Show A has trace zero iff it is the sum of 2 nilpotent matrices.

(5 points)

Q2 Show that if an upper traingular matrix is diagonalizable, then it is diagonalizable by an upper triangular similarity. (Hint: First prove that every square matrix is factorizable as RQ, where R is upper triangular and Q unitary)

(5 points)

Q3 Let $A, B \in M_n$ and C their commutator. Suppose C commutes with both A and B. Then it is known (e.g., Jacobson's Lemma) that C is nilpotent. Show that C^{n-1} is already zero. (Hint: If $C^n = 0$ but $C^{n-1} \neq 0$, then why does it follow that both A and B are polynomials in C?)

(5 points)

Q4 Show that A is normal iff the singular values of A coincide with the absolute values of the eigenvalues of A.

(5 points)