Name: ______________________________

Qualifying Exam, April 2012
Real Analysis I
Dr. Yan Cao

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (20 points.)
Let $\mathbb{R}$ be the set of real numbers. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$. Let $\mathcal{E} = \{[a, \infty) : a \in \mathbb{R}\}$. Let $\mathcal{M}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E}$. Give a complete proof that $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$.

Problem 2 (20 points.)
Let m^* denote the Lebesgue outer measure on $\mathbb{R}$, the set of real numbers.
(a) State the definition of Lebesgue outer measure m^* on $\mathbb{R}$.
(b) Show that m^* is countably subadditive, i.e. $m^*(\cup_1^\infty A_j) \leq \sum_{j=1}^\infty m^*(A_j)$, where A_j's are any subsets of $\mathbb{R}$.
(c) Show that $m^*(\mathbb{Q}) = 0$, where $\mathbb{Q}$ is the set of rational numbers.

Problem 3 (20 points.)
Let $f : \mathbb{R} \to [0, \infty]$ be a measurable function. Prove that $\int_{\mathbb{R}} f = 0$ if and only if $f = 0$ a.e.

Problem 4 (20 points.)
Let $f : [0, 1] \to \mathbb{R}$ be continuous. Let $g : [0, 1] \to \mathbb{R}$ be Lebesgue measurable and $0 \leq g(x) \leq 1$ for $x \in [0, 1]$. Find the limit

$$\lim_{n \to \infty} \int_0^1 f((g(x))^n)dx.$$

Problem 5 (20 points.)
Let m be the Lebesgue measure. Let A be a measurable subset of $\mathbb{R}$ with $0 < m(A) < \infty$. Show that for $1 < p < q \leq \infty$, we have $L^q(A) \subseteq L^p(A)$.

Name: ______________________________

Qualifying Exam, April 2012
Real Analysis II
Dr. Yan Cao

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (20 points.)
Let c_0 be the space of all sequences of real numbers converging to zero. Show that the dual (conjugate) space of c_0 is l_1.

Problem 2 (20 Points.)
Let H be a Hilbert space. Let $f : H \to \mathbb{R}$ be a bounded linear functional and $f \neq 0$. Let N be the null space (kernel) of f. Let

$$N^{\perp} = \{y \in H| < y, x >= 0 \ \forall x \in N\}. \tag{1}$$

(a) Show that N is a closed subspace of H.
(b) Show that the dimension of $N^{\perp}$ is 1.
(c) Show that there is exactly one $x_0 \in H$, such that

$$f(x) =< x, x_0 >, \text{ and } \|f\| = \|x_0\|. \tag{2}$$

Problem 3 (20 points.)
Let H be a Hilbert space, and (e_k) be an orthonormal sequence in H. Then:
(a) The series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges (in the norm of H) if and only if the series $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.
(b) If $\sum_{k=1}^{\infty} \alpha_k e_k$ converges to x , then $\alpha_k =< x, e_k >$; hence $x = \sum_{k=1}^{\infty} < x, e_k > e_k$.

Problem 4 (20 Points.)
Let (T_n) be a sequence of bounded self-adjoint linear operators $T_n : H \to H$ on a Hilbert space H. Assume that (T_n) converges to T, i.e. $\|T_n - T\| \to 0$, where $\|\cdot\|$ is the norm on $B(H, H)$, the space of all bounded linear operators from H into H. Show that
(a) T is linear.
(b) T is bounded.
(c) T is self-adjoint.
(d) $< Tx, x >$ is real for all $x \in H$.

Problem 5 (20 Points.)
Let X and Y be Banach spaces. Let $B(X, Y)$ be the space of all bounded linear operators from X into Y. Assume $T_n \in B(X, Y)$, $n = 1, 2, 3, ...$ Show that the following statements are equivalent:
(a) $(\|T_n\|)$ is bounded,
(b) $(\|T_n x\|)$ is bounded for every fixed $x \in X$,
(c) $(|g(T_n x)|)$ is bounded for every fixed $x \in X$ and every fixed $g \in Y'$, where Y' is the dual (conjugate) space of Y.

Name: ________________________________

Abstract Algebra Qualifying Exam
Spring 2012
Tobias Hagge

April 10, 2012

Solve any five problems. Show all work.

1. Let M be a set with two binary operations $\cdot : M \times M \to M$ and $\star : M \times M \to M$ such that both $(M, \cdot)$ and $(M, \star)$ are monoids. Suppose $\cdot$ is a monoid homomorphism $\cdot : (M, \star) \times (M, \star) \to (M, \star)$ and similarly $\star$ is a monoid homomorphism $\star : (M, \cdot) \times (M, \cdot) \to (M, \cdot)$. Show that
    1. $(M, \cdot) = (M, \star)$,
    2. $(M, \cdot)$ is commutative.
2. Classify all transitive G-sets up to G-set isomorphism, for the case $G = A_4$.
3. Show that a group of order 56 cannot be simple.
4. Show that every group of order p^2, with p prime, is abelian.
5. Prove that every symmetric group S_n has an index 2 subgroup.
6. Let R be a commutative ring with 1, M a maximal ideal. Show that R/M is a field.
7. Show that every finitely generated Euclidean domain is a principal ideal domain.

Qualifying Exam

Math 6315, Ordinary Differential Equations I, April 2012

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 34 points each

Give clear and complete answers with full details in proofs

1. Consider the n dimensional system

$$x' = Ax + f(x)$$

where A is a constant martix with $|e^{At}| \leq d^{-9t}$ for all $t \geq 0$. Suppose that $f(x)$ is continuously differentiable and satisfies $|f(x)| \leq 5|x|$ for all $x \epsilon \mathbf{R}^n$. Show that the solution of

$$x' = Ax + f(x), \quad x(0) = \xi$$

goes to zero as $t \longrightarrow \infty$

2. Consider the initial value problem

$$x' = -x + t, \; x(0) = \xi$$

Find the solution by using the method of sussessive approximations.

3. Consider the boundary value problem

$$y'' + \lambda y = 0, \quad 0 \leq t \leq 1, \quad y(0) = 0, \; y(1) = 0$$

Is $\lambda = 0$ an eigenvalue? Compute the Green's function $G(t, s, 0)$ for $0 \leq s, t \leq 1$.

Qualifying Exam
Math 6316, Ordinary Differential Equations II, April 2012

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 34 points each
Give clear and complete answers with full details in proofs

1. Determine the stability of the trivial solution of

$$\begin{aligned} x_1' &= x_2 + x_1^2 x_2 \\ x_2' &= x_2 - x_1 + x_2^2 x_1 \end{aligned}$$

by finding an appropriate Lyapunov function. You must prove that the function you find is actually a Lyapunov function for this system.

2. Find the stable and unstable manifolds of the system

$$\begin{aligned} x_1' &= x_1 + x_2^3 \\ x_2' &= -x_2 \end{aligned}$$

Hint: Try a coordinate change (in which the first new coordinate involves x_1 and x_2^3) that maps the system to its linear part.

3. Consider the system

$$x' = Ax + F(x)$$

where A is an $n \times n$ constant matrix and $F : \mathbf{R}^2 \to \mathbf{R}^2$ is continuous. Suppose $F(x) = o(|x|)$ as $|x| \to 0$ and there exist constants N and α such that

$$|e^{At}| \leq Ne^{-\alpha t}$$

for all $t \geq 0$. Prove that if $0 < \beta < \alpha$ and if $|\phi(0)|$ is sufficiently small, then

$$|\phi(t)| \leq N|\phi(0)|e^{-(\alpha-\beta)t}$$

for all $t \geq 0$.

Name: ______________________

Complex Analysis Qualifying Exam
Spring 2012
Tobias Hagge

April 12, 2012

$B_r(z)$ denotes the open ball of radius r, centered at z.

1. Let $a \in \mathbb{C} \cup \infty$ such that $|a| \neq 1$. Show that the circle through a, $\bar{a}^{-1}$, and 1 intersects the unit circle at right angles.
2. Let $D = B_1(0)$, $f : \bar{D} \to \bar{D}$ analytic. Show that if f has a zero of order $n > 0$ at $z = 0$, then $|f(z)| \leq |z|^n$ on $\bar{D}$.
3. Show that a nowhere zero, nonconstant entire function has an essential singularity at infinity.
4. Let $D_1 = B_1(0)$, $D_2 = B_2(0)$. Let $A = D_2 - \overline{D_1}$. Suppose f is analytic on $\overline{A}$. Show that f may be written as $g - h$, where g is analytic on $\overline{D}_2$ and h is analytic on $\mathbb{C} - D_1$.
5. Compute the residue at $z = 0$ of the function $\frac{e^z}{\sin^3 z}$.
6. Compute $\int_0^\infty \frac{\cos x}{x^3+x} dx$ by complex methods.

Topology
(Dr. Mieczyslaw K. Dabkowski)
Qualifying Exam
Friday, April 13, 2012

Name________________________

Instructions. *Please solve any* **four problems** *from the list of the following problems (show all your work).*

1. Let $X = \mathbb{R}^2$. Check if the following function $d : X \times X \to \mathbb{R}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|,\ |x_2 - y_2|\}$$

is a metric on X.

2. Let G be an open subset of X and $A \subset X$.

a) Prove that if $G \cap \overline{A} \neq \emptyset$, then $G \cap A \neq \emptyset$.

b) Prove that $\overline{G \cap \overline{A}} = \overline{G \cap A}$.

c) Show by example that the condition for G to be open is necessary.

3. Let $\mathbb{R}_\tau$ be the set of real numbers with topology $\tau = \{(-x, x) \mid x > 0\} \cup \{\emptyset, \mathbb{R}\}$ and $\mathbb{R}_\tau \times \mathbb{R}_\tau$ be the product topology on $\mathbb{R}^2$.

a) Prove that $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is open in $\mathbb{R}_\tau \times \mathbb{R}_\tau$.

b) Find $\overline{A}$. (justify briefly your answer)

4. Show that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in $X \times X$.

5. Let A be a countable subset of $\mathbb{R}^n$, where $n > 1$. Show that $\mathbb{R}^n \backslash A$ is path-connected.

6. Let $\mathbb{C}$ be the set of complex numbers. Consider the finite complement topology $\tau = \{A \subset \mathbb{C} \mid \mathbb{C} \backslash A$ is finite$\} \cup \{\emptyset\}$.

a) Is $(\mathbb{C}, \tau)$ T_0? Yes or No (justify your answer).

b) Is $(\mathbb{C}, \tau)$ T_1? Yes or No (justify your answer).

c) Is $(\mathbb{C}, \tau)$ T_2? Yes or No (justify your answer).

d) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial function, that is $f(z) = \sum_{i=0}^{n} a_i z^i$, where $a_i \in \mathbb{C}$ for $i = 0, 1, 2, ..., n$. Is f a continuous map from $(\mathbb{C}, \tau)$ to itself? Justify briefly your answer.

e) Let $g : \mathbb{C} \longrightarrow \mathbb{C}$, $g(z) = e^z$, answer the same question as in point **d**).

f) If we consider the restriction of g to $\mathbb{R}$ (the set of real numbers), is g continuous?