REAL ANALYSIS (Dr. Mieczyslaw K. Dabkowski) Qualifying Exam April 11, 2011

Name\_

Instructions. Please solve any five problems from the list of the following problems (show all your work).

1. Show that the following family of subsets of  $\mathbb{R}$ :

 $\mathcal{M} = \{ E \subset \mathbb{R} \mid E \text{ is countable or } E^c \text{ is countable} \}$ 

is a  $\sigma$ - algebra in  $\mathbb{R}$ .

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2. Recall, the following definition of measurable function:

**Definition** Let M is a  $\sigma$ -algebra in X. A function  $h : X \to [-\infty, \infty]$  is measurable if the set  $\{x \in X \mid h(x) \ge r\}$  is measurable for every  $r \in \mathbb{R}$ .

Suppose that  $f, g : \mathbb{R} \to [-\infty, \infty]$  are measurable functions. Using the definition stated above, show that the following set

$$\{x \in \mathbb{R} \mid f(x) < g(x)\}\$$

is measurable.

3. Suppose that  $f_n : \mathbb{R} \to [0, \infty]$  is measurable for n = 1, 2, 3, ..., and  $f_1 \ge f_2 \ge ... \ge 0$ ,  $f_n(x) \to f(x)$  as  $n \to \infty$ , for every  $x \in \mathbb{R}$ , and  $f_1 \in L^1(\mu)$ , where  $\mu$  is the Lebesgue measure. Prove that then

$$\lim_{n\to\infty}\int_{\mathbb{R}}f_n\ d\mu=\int_{\mathbb{R}}f\ d\mu$$

and show that this conclusion does not follow if the condition " $f_1 \in L^1(\mu)$ " is omitted.

4. Suppose  $\mu(X) = 1$  and suppose f and g are positive measurable functions on X such that  $fg \ge 1$ . Prove that

$$\left(\int_X f \ d\mu\right) \left(\int_X g \ d\mu\right) \ge 1.$$

5. Let  $\mu$  is the Lebesgue measure on  $\mathbb{R}$ . Suppose  $f \in L^1(\mu)$ . Prove that to each  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$\int_E |f| \ d\mu < \epsilon$$

whenever  $\mu(E) < \delta$ .

6. Suppose  $f : \mathbb{R} \to \mathbb{R}$  is a measurable function on  $\mathbb{R}$  and let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$ . Define the following function

$$\varphi(p) = \int_{\mathbb{R}} |f|^p d\mu = ||f||_p^p, \ (0$$

Let  $E = \{p \mid \varphi(p) < \infty\}$  and assume that  $|| f ||_{\infty} > 0$ . If  $r , and <math>s \in E$ , prove that  $p \in E$ .

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## Math 6302 Qualifying Exam 2011

1) Show that if a normed space X has the property that the closed unit ball is compact , then X is finite dimensional.

2) Let  $T: X \to Y$  be a linear operator and dim  $X = \dim Y = n < \infty$ . Show that R(T) = Y if and only if  $T^{-1}$  exists.

3)(5 pts.) Let X be a Banach space, Y a normed space and  $T_n \in B(X, Y)$  such that  $(T_n x)$  is Cauchy in Y for every  $x \in X$ . Show that  $(||T_n||)$  is bounded.

4) If  $x_n \in C[a, b]$  and  $(x_n)$  converges weakly to  $x \in C[a, b]$ , show that  $(x_n)$  is pointwise convergent on [a, b].

5) If  $T : X \to Y$  is a closed linear operator, where X and Y are normed spaces and Y is compact, show that T is bounded.

#### Qualifying Exam: Ordinary Differential Equations I, April 2011

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM Problems count 34 points each Give clear and complete answers with full details in proofs

- 1. Prove that if  $\Phi(t)$  is a fundamental set of solutions of x' = A(t)x with periodic coefficient A(t) = A(t+T), then  $\Phi(t+T)$  is also a fundamental set of solutions of x' = A(t)x. Furthermore exist a non-singular periodic matrix P(t), with period T and a constant matrix R such that  $\Phi(t) = P(t)e^{tR}$ .
- 2. If  $a_1$  and  $a_2$  are constants, find a fundamental set of solutions of

$$t^2 y'' + a_1 t y' + a_2 y = 0 \qquad 0 < t < \infty$$

*Hint.* Use the change of variables  $x = \ln t$ .

3. Find the solution to the following IVP for  $y_1(0) = 1$ ,  $y_2(0) = 1$  and  $t \ge 0$ ,

$$y'_1 = y_1 + y_2 + e^t$$
  
 $y'_2 = y_1 - y_2 + 1$ 

by first finding a fundamental set of solutions to homogeneous version of above system of differential equations; using the Jordan canonical form.

Fri 4/15 From skyan Dr. Muskyan

# Qualifying Exam Problems: Algebra

Choose 4 problems only.

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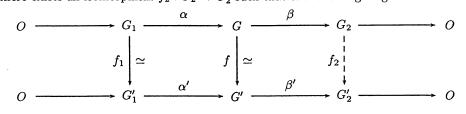
Question #1: Assume that  $G_1$ , G,  $G_2$  are groups (not necessarily Abelian) and denote by O the trivial group (composed of a single element). The sequence of homomorphisms

 $O \xrightarrow{\alpha} G_1 \xrightarrow{\alpha} G \xrightarrow{\beta} G_2 \xrightarrow{\beta} O$ 

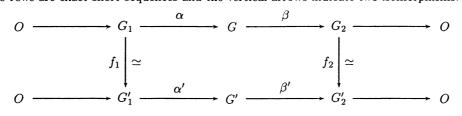
is called an *exact short sequence* if  $\alpha$  is monomorphism,  $\beta$  is epimorphism and Im  $\alpha = \text{Ker }\beta$ . Prove or give a counterexample for the following statements:

(a) Assume that we have the following commutative diagram of groups and homomorphisms where the two rows are exact short sequences and the vertical arrows indicate two isomorphisms:

Then there exists an isomorphism  $f_2: G_2 \to G'_2$  such that the following diagram commutes:

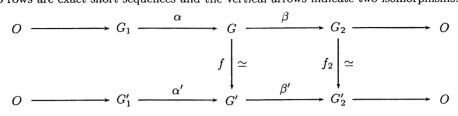


(b) Assume that we have the following commutative diagram of groups and homomorphisms where the two rows are exact short sequences and the vertical arrows indicate two isomorphisms:



Then there exists an isomorphism  $f: G \to G'$  such that the following diagram commutes:

(c) Assume that we have the following commutative diagram of groups and homomorphisms where the two rows are exact short sequences and the vertical arrows indicate two isomorphisms:



Then there exists an isomorphism  $f_1: G_1 \to G'_1$  such that the following diagram commutes:

Question #2: Consider the group  $S_4$  of permutations of four symbols. (a) List all the elements of  $S_4$  (using the symbols of cycles, e.g. (1,2)(3,4)) arranged in the conjugacy classes.

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(b) It is well known that the elements of  $S_4$  can be identified with the (rigid) symmetries of a solid cube. Explain how this identification is done.

(c) Clearly the dihedrall group  $D_4$  of symmetries of a square is a subgroup of the group of symmetries of the cube. Identify this subgroup in  $S_4$  by listing all its elements.

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(d) Clearly the dihedral group  $D_3$  of symmetries of a triangle is a subgroup of the group of symmetries of the cube. Identify this subgroup in  $S_4$  by listing all its elements.

(e) The cube contains an inscribed tetrahedron, therefore the symmetries of the tetrahedron (i.e. the so-called tetrahedral group  $\mathbb{T}$ ) is a subgroup of  $S_4$ . List all the elements of this subgroup.

Question #3: Consider the group  $S_4$  of permutations of four symbols. The so-called Klein's subgroup  $V_4$  of  $S_4$  consists of four elements

$$V_4 := \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

(a) What is the normalizer  $N(V_4)$  of  $V_4$  in  $S_4$ ?

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(b) Compute the Weyl group  $W(V_4)$  of  $V_4$  in  $S_4$ . Can you identify this group as one of the wellknown classical groups? (Hint: the Weyl group W(H) of  $H \in G$  is  $N_G(H)/H$ , where  $N_H(H)$ stands for the normalizer of H in G). \*

(c) Find all the normal subgroups H of  $S_4$  and compute the quotient groups  $S_4/H$  (i.e. identify this quotient groups as classical groups).

Question #4: Let G be a group. Denote by Aut(G) the group of all automorphisms of G and by Inn(G) the subgroup of all inner automorphisms of G. Show that Inn(G) is a normal subgroup of Aut(G).

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Question #5: Identify the group  $Inn(D_3)$  as one of the classical groups.

#### COMPLEX ANALYSIS - APRIL 2011 QUALIFYING EXAMINATION

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TOBIAS HAGGE

Do as many problems as you are able. Your lowest scoring answer will not count toward your score. Show work and justifications clearly.

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(1) Compute  $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1} dx$  using complex methods.

(2) Compute  $\lim_{k\to\infty} \int_{-k}^{k} \frac{dx}{x+i}$  for large k using residue calculus methods. Does  $\int_{-\infty}^{\infty} \frac{dx}{x+i}$  exist?

(3) Let f be analytic at a point a. State and prove Cauchy's Estimate for  $|f^{(n)}(z)|$ .

(4) Let  $P = \sum_{i=0}^{n} a_i z^i$  be a polynomial such that  $|a_i| = 1$  for all  $i \in [0 \dots n]$ . State Rouche's Theorem and use it to show that  $P(z) = 0 \Rightarrow \frac{1}{2} \le |z| \le 2$ .

(5) Define the algebraic order of an analytic function f at the point a. Show that it is always an integer. (Hint: Use Taylor expansion at a).

- (6) (a) Show that an injective entire function cannot have an essential singularity at  $\infty$ .
  - (b) Show that such a function is of the form  $f(z) = k(z z_0)$  where k and  $z_0$  are constants.

(7) Let D be the unit disk centered at the origin. Prove that every analytic bijection  $f: D \to D$  is a rotation.

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### Qualifying Exam: Ordinary Differential Equations II, April 2011

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM Problems count 34 points each Give clear and complete answers with full details in proofs

1. Determine the stability of x = 0 for the nonlinear system

$$x'_{1} = -x_{2} + 2\sin(3x_{1})$$
$$x'_{2} = x_{1} - x_{1}x_{2}^{2}$$

by finding an appropriate Lyapunov function.

2. Determine the stability of the trivial solution of the system

$$\begin{aligned} x_1' &= x_2 - x_1 x_2 x_3 \\ x_2' &= x_3 - x_2 x_3^2 \\ x_3' &= -2x_1 - 5x_2 - 4x_3 + x_1^2 x_2 x_3^4 \end{aligned}$$

$$\begin{aligned} x_1' &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\ x_2' &= x_1 + x_2(1 - x_1^2 - x_2^2) \end{aligned}$$

have a periodic solution? If so, is it orbitally stable? Hint: It may be helpful to look at the problem in polar coordinates.

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# **Examination Booklet**

# MATH 6390: Introduction to Nonlinear Analysis

QUAL Exam, April 18, 2011

Last Name:

(print)

First Name:

(print)

Signature:

Read all of the following information before starting the exam:

- There are four problems in this exam.
- Show the significant steps of your work clearly for the problems.
- Circle or otherwise indicate your final answers.
- If you need to visit the restroom, bring your paper to the proctor.
- You may not leave the exam until 30 minutes have elapsed.
- Good luck!

Question	Weight	Your score
1	30	
2	30	
3	30	
4	10	
Total		

## For Instructor's Use Only

## 1. Does the system

$$\begin{cases} 3x^2 + txy + y^2 = 0\\ x^2 + 3xy + 3y^2 = 0\\ x^2 - 2x + y^2 + 4y = 15 \end{cases}$$

have a solution? Argue.

### 2. Does the differential system

$$\dot{z} = \frac{z^3}{(z-1)^4(z+i)^3} \qquad (z \in \mathbb{C})$$

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admit a periodic solution? Argue.

3. Let  $S \subset \mathbb{R}^2$  be the square with the vertices (-1, -1), (-1, 1), (1, -1), (1, 1). Let  $\sigma, \alpha : [-1, 1] \to S$  be two continuous curves satisfying the conditions:  $\sigma(-1) = (-1, -1), \sigma(1) = (1, 1), \alpha(-1) = (-1, 1), \alpha(1) = (1, -1)$  Show that these curves intersect.

4. Let  $f: \mathbb{R}^2 \to \mathbb{R}^2$  be a continuous map such that

$$\lim_{|x|\to\infty}\frac{(f(x),x)}{|x|}=\infty$$

(here " $(\cdot, \cdot)$ " stands for the inner product). Show that f is surjective.

Hint: Given  $y \in \mathbb{R}^n$ , consider the equation f(x) = y, take a big ball and use a homotopy to a map of degree 1.