REAL ANALYSIS
(Dr. Mieczyslaw K. Dabkowski)
Qualifying Exam
April 11, 2011

Name_______________________

Instructions. *Please solve any* **five problems** *from the list of the following problems (show all your work).*

1. Show that the following family of subsets of $\mathbb{R}$:
$$\mathcal{M} = \{E \subset \mathbb{R} \mid E \text{ is countable or } E^c \text{ is countable}\}$$
is a $\sigma-$ algebra in $\mathbb{R}$.

2. Recall, the following definition of measurable function:
Definition *Let M is a σ-algebra in X. A function $h : X \to [-\infty, \infty]$ is measurable if the set $\{x \in X \mid h(x) \geq r\}$ is measurable for every $r \in \mathbb{R}$.*
Suppose that f, $g : \mathbb{R} \to [-\infty, \infty]$ are measurable functions. Using the definition stated above, show that the following set
$$\{x \in \mathbb{R} \mid f(x) < g(x)\}$$
is measurable.

3. Suppose that $f_n : \mathbb{R} \to [0, \infty]$ is measurable for $n = 1, 2, 3, ...$, and $f_1 \geq f_2 \geq ... \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in \mathbb{R}$, and $f_1 \in L^1(\mu)$, where μ is the Lebesgue measure. Prove that then
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu$$
and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

4. Suppose $\mu(X) = 1$ and suppose f and g are positive measurable functions on X such that $fg \geq 1$. Prove that
$$\left(\int_X f \, d\mu\right)\left(\int_X g \, d\mu\right) \geq 1.$$

5. Let μ is the Lebesgue measure on $\mathbb{R}$. Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that
$$\int_E |f| \, d\mu < \epsilon$$
whenever $\mu(E) < \delta$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a measurable function on $\mathbb{R}$ and let μ be the Lebesgue measure on $\mathbb{R}$. Define the following function
$$\varphi(p) = \int_{\mathbb{R}} | f |^p \, d\mu = \| f \|_p^p, \ (0 < p < \infty).$$
Let $E = \{p \mid \varphi(p) < \infty\}$ and assume that $\| f \|_\infty > 0$. If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

Math 6302 Qualifying Exam 2011

1) Show that if a normed space X has the property that the closed unit ball is compact , then X is finite dimensional.

2) Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $R(T) = Y$ if and only if T^{-1} exists.

3)(5 pts.) Let X be a Banach space, Y a normed space and $T_n \in B(X, Y)$ such that $(T_n x)$ is Cauchy in Y for every $x \in X$. Show that $(\|T_n\|)$ is bounded.

4)If $x_n \in C[a, b]$ and (x_n) converges weakly to $x \in C[a, b]$, show that (x_n) is pointwise convergent on $[a, b]$.

5) If $T : X \to Y$ is a closed linear operator, where X and Y are normed spaces and Y is compact, show that T is bounded.

Qualifying Exam: Ordinary Differential Equations I, April 2011

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 34 points each
Give clear and complete answers with full details in proofs

1. Prove that if $\Phi(t)$ is a fundamental set of solutions of $x' = A(t)x$ with periodic coefficient $A(t) = A(t+T)$, then $\Phi(t+T)$ is also a fundamental set of solutions of $x' = A(t)x$. Furthermore exist a non-singular periodic matrix $P(t)$, with period T and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$.

2. If a_1 and a_2 are constants, find a fundamental set of solutions of

$$t^2y'' + a_1ty' + a_2y = 0 \qquad 0 < t < \infty$$

Hint. Use the change of variables $x = \ln t$.

3. Find the solution to the following IVP for $y_1(0) = 1,\ y_2(0) = 1$ and $t \geq 0$,

$$\begin{aligned} y_1' &= y_1 + y_2 + e^t \\ y_2' &= y_1 - y_2 + 1 \end{aligned}$$

by first finding a fundamental set of solutions to homogeneous version of above system of differential equations; using the Jordan canonical form.

Fri 4/15
From Dr. Hooshyar

Qualifying Exam Problems: Algebra

Choose 4 problems only.

NAME: ______________________________

Question #1: Assume that G_1, G, G_2 are groups (not necessarily Abelian) and denote by O the trivial group (composed of a single element). The sequence of homomorphisms

$$O \longrightarrow G_1 \xrightarrow{\ \alpha\ } G \xrightarrow{\ \beta\ } G_2 \longrightarrow O$$

is called an *exact short sequence* if α is monomorphism, β is epimorphism and $\operatorname{Im}\alpha = \operatorname{Ker}\beta$. Prove or give a counterexample for the following statements:

(a) Assume that we have the following commutative diagram of groups and homomorphisms where the two rows are exact short sequences and the vertical arrows indicate two isomorphisms:

$$\begin{array}{ccccccccc}
O & \longrightarrow & G_1 & \xrightarrow{\ \alpha\ } & G & \xrightarrow{\ \beta\ } & G_2 & \longrightarrow & O \\
 & & f_1 \Big\downarrow \simeq & & f \Big\downarrow \simeq & & & & \\
O & \longrightarrow & G_1' & \xrightarrow{\ \alpha'\ } & G' & \xrightarrow{\ \beta'\ } & G_2' & \longrightarrow & O
\end{array}$$

Then there exists an isomorphism $f_2 : G_2 \to G_2'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
O & \longrightarrow & G_1 & \xrightarrow{\ \alpha\ } & G & \xrightarrow{\ \beta\ } & G_2 & \longrightarrow & O \\
 & & f_1 \Big\downarrow \simeq & & f \Big\downarrow \simeq & & f_2 \Big\downarrow & & \\
O & \longrightarrow & G_1' & \xrightarrow{\ \alpha'\ } & G' & \xrightarrow{\ \beta'\ } & G_2' & \longrightarrow & O
\end{array}$$

(b) Assume that we have the following commutative diagram of groups and homomorphisms where the two rows are exact short sequences and the vertical arrows indicate two isomorphisms:

$$\begin{array}{ccccccccc} O & \longrightarrow & G_1 & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & G_2 & \longrightarrow & O \\ & & f_1 \big\downarrow \simeq & & & & f_2 \big\downarrow \simeq & & \\ O & \longrightarrow & G_1' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & G_2' & \longrightarrow & O \end{array}$$

Then there exists an isomorphism $f : G \to G'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} O & \longrightarrow & G_1 & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & G_2 & \longrightarrow & O \\ & & f_1 \big\downarrow \simeq & & f \,\vdots\!\downarrow & & f_2 \big\downarrow \simeq & & \\ O & \longrightarrow & G_1' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & G_2' & \longrightarrow & O \end{array}$$

(c) Assume that we have the following commutative diagram of groups and homomorphisms where the two rows are exact short sequences and the vertical arrows indicate two isomorphisms:

$$
\begin{CD}
O @>>> G_1 @>\alpha>> G @>\beta>> G_2 @>>> O \\
@. @. @V{f}V{\simeq}V @V{f_2}V{\simeq}V @. \\
O @>>> G_1' @>\alpha'>> G' @>\beta'>> G_2' @>>> O
\end{CD}
$$

Then there exists an isomorphism $f_1 : G_1 \to G_1'$ such that the following diagram commutes:

$$
\begin{CD}
O @>>> G_1 @>\alpha>> G @>\beta>> G_2 @>>> O \\
@. @V{f_1}VV @V{f}V{\simeq}V @V{f_2}V{\simeq}V @. \\
O @>>> G_1' @>\alpha'>> G' @>\beta'>> G_2' @>>> O
\end{CD}
$$

Question #2: Consider the group S_4 of permutations of four symbols.
(a) List all the elements of S_4 (using the symbols of cycles, e.g. $(1,2)(3,4)$) arranged in the conjugacy classes.

(b) It is well known that the elements of S_4 can be identified with the (rigid) symmetries of a solid cube. Explain how this identification is done.

(c) Clearly the dihedrall group D_4 of symmetries of a square is a subgroup of the group of symmetries of the cube. Identify this subgroup in S_4 by listing all its elements.

(d) Clearly the dihedral group D_3 of symmetries of a triangle is a subgroup of the group of symmetries of the cube. Identify this subgroup in S_4 by listing all its elements.

(e) The cube contains an inscribed tetrahedron, therefore the symmetries of the tetrahedron (i.e. the so-called tetrahedral group $\mathbb{T}$) is a subgroup of S_4. List all the elements of this subgroup.

Question #3: Consider the group S_4 of permutations of four symbols. The so-called Klein's subgroup V_4 of S_4 consists of four elements

$$V_4 := \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

(a) What is the normalizer $N(V_4)$ of V_4 in S_4?

(b) Compute the Weyl group $W(V_4)$ of V_4 in S_4. Can you identify this group as one of the well-known classical groups? (Hint: the Weyl group $W(H)$ of $H \in G$ is $N_G(H)/H$, where $N_H(H)$ stands for the normalizer of H in G). *

(c) Find all the normal subgroups H of S_4 and compute the quotient groups S_4/H (i.e. identify this quotient groups as classical groups).

Question #4: Let G be a group. Denote by $\text{Aut}(G)$ the group of all automorphisms of G and by $\text{Inn}(G)$ the subgroup of all inner automorphisms of G. Show that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$.

Question #5: Identify the group $\mathrm{Inn}(D_3)$ as one of the classical groups.

COMPLEX ANALYSIS - APRIL 2011
QUALIFYING EXAMINATION

TOBIAS HAGGE

Do as many problems as you are able. Your lowest scoring answer will not count toward your score. Show work and justifications clearly.

(1) Compute $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx$ using complex methods.

(2) Compute $\lim_{k\to\infty} \int_{-k}^{k} \frac{dx}{x+i}$ for large k using residue calculus methods. Does $\int_{-\infty}^{\infty} \frac{dx}{x+i}$ exist?

(3) Let f be analytic at a point a. State and prove Cauchy's Estimate for $|f^{(n)}(z)|$.

(4) Let $P = \sum_{i=0}^{n} a_i z^i$ be a polynomial such that $|a_i| = 1$ for all $i \in [0 \ldots n]$. State Rouche's Theorem and use it to show that $P(z) = 0 \Rightarrow \frac{1}{2} \leq |z| \leq 2$.

(5) Define the algebraic order of an analytic function f at the point a. Show that it is always an integer. (Hint: Use Taylor expansion at a).

(6) (a) Show that an injective entire function cannot have an essential singularity at ∞.
(b) Show that such a function is of the form $f(z) = k(z - z_0)$ where k and z_0 are constants.

(7) Let D be the unit disk centered at the origin. Prove that every analytic bijection $f : D \to D$ is a rotation.

"Choice"

Qualifying Exam: Ordinary Differential Equations II, April 2011

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 34 points each
Give clear and complete answers with full details in proofs

1. Determine the stability of $x = 0$ for the nonlinear system
$$x_1' = -x_2 + 2\sin(3x_1)$$
$$x_2' = x_1 - x_1 x_2^2$$
by finding an appropriate Lyapunov function.

2. Determine the stability of the trivial solution of the system
$$x_1' = x_2 - x_1 x_2 x_3$$
$$x_2' = x_3 - x_2 x_3^2$$
$$x_3' = -2x_1 - 5x_2 - 4x_3 + x_1^2 x_2 x_3^4$$

3. Does the following 2 dimensional system
$$x_1' = -x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$x_2' = x_1 + x_2(1 - x_1^2 - x_2^2)$$
have a periodic solution? If so, is it orbitally stable? Hint: It may be helpful to look at the problem in polar coordinates.

"Choice"

Examination Booklet

MATH 6390: Introduction to Nonlinear Analysis

QUAL Exam, April 18, 2011

Last Name: ____________________ (print)

First Name: ____________________ (print)

Signature: ____________________

Read all of the following information before starting the exam:

- There are **four** problems in this exam.
- Show the significant steps of your work clearly for the problems.
- Circle or otherwise indicate your final answers.
- If you need to visit the restroom, bring your paper to the proctor.
- You may not leave the exam until 30 minutes have elapsed.
- Good luck!

For Instructor's Use Only

Question	Weight	Your score
1	30	
2	30	
3	30	
4	10	
	Total	

1. Does the system

$$\begin{cases} 3x^2 + txy + y^2 = 0 \\ x^2 + 3xy + 3y^2 = 0 \\ x^2 - 2x + y^2 + 4y = 15 \end{cases}$$

have a solution? Argue.

2. Does the differential system

$$\dot{z} = \frac{z^3}{(z-1)^4(z+i)^3} \qquad (z \in \mathbb{C})$$

admit a periodic solution? Argue.

3. Let $S \subset \mathbb{R}^2$ be the square with the vertices $(-1,-1), (-1,1), (1,-1), (1,1)$. Let $\sigma, \alpha : [-1,1] \to S$ be two continuous curves satisfying the conditions: $\sigma(-1) = (-1,-1), \sigma(1) = (1,1), \alpha(-1) = (-1,1), \alpha(1) = (1,-1)$ Show that these curves intersect.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map such that

$$\lim_{|x| \to \infty} \frac{(f(x), x)}{|x|} = \infty$$

(here "$(\cdot, \cdot)$" stands for the inner product). Show that f is surjective.

Hint: Given $y \in \mathbb{R}^n$, consider the equation $f(x) = y$, take a big ball and use a homotopy to a map of degree 1.