Name: ______________________________

Qualifying Exam, April 2010
Real Analysis I

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Solve Problem 1- 4 and one of Problem 5 and 6.

Problem 1 (20 points.)

Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure, i.e. $\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_{j=1}^{\infty} A_j\}$.

a. For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

b. There exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B) = \mu^*(E)$.

Problem 2 (20 points.)

Let $(X, \mathcal{M})$ be a measurable space and μ be a measure on it. Suppose $f : X \to [0, \infty]$ is measurable on $(X, \mathcal{M})$. Define $\nu(E) = \int_E f d\mu$ for any $E \in \mathcal{M}$. Show that ν is a measure.

Problem 3 (20 points.)

Let $(X, \mathcal{M})$ be a measurable space. Let $\{f_n\}$ be a sequence of real-valued measurable functions on $(X, \mathcal{M})$. Prove that $\limsup_{n\to\infty} f_n(x)$ is measurable.

Problem 4 (20 points.)

Let $(X, \mathcal{M})$ be a measurable space and μ be a measure on it. Let $\{f_n\}$ be a sequence of positive measurable functions on $(X, \mathcal{M})$. Assume that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$, and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu < \infty$. Prove that $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$ for each $E \in \mathcal{M}$.

Problem 5 (20 points.)

Let f be a Lebesgue measurable function on $[0, 1]$ and $0 < p < q \leq \infty$. Assume that $f \in L^q[0, 1]$. Show that $f \in L^p[0, 1]$ and $\|f\|_p \leq \|f\|_q$.

Problem 6 (20 points.)

Suppose $\{g_k(x)\}$ is a sequence of absolutely continuous functions on $[a, b]$. And there is a function $F \in L^1[a, b]$, such that $|g_k'(x)| \leq F(x)$ a.e. for all $k \in \mathbb{N}$. Also assume that $\lim_{k\to\infty} g_k(x) = g(x)$ and $\lim_{k\to\infty} g_k'(x) = f(x)$ a.e. Prove that $g'(x) = f(x)$ a.e.

Name: ______________________________

Qualifying Exam, April 2010
Real Analysis II

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (20 points.)
Let $X = C[0,1]$, the set of all continuous real-valued functions on $J = [0,1]$, and let $d(x,y) = \max_{t\in J}|x(t) - y(t)|$. Show that this metric space (X,d) is complete.

Problem 2 (25 points.)
Let $C[0,1]$ denote the normed space of all continuous real-valued functions on $[0,1]$, and let $C^1[0,1]$ denote the normed space of all differentiable real-valued functions whose derivatives are continuous on $[0,1]$. The norms in both space are defined by $||x|| = \max_{s\in[0,1]}|x(s)|$.
(a) Define $D : C^1[0,1] \to C[0,1]$ by $(Dx)(s) = x'(s)$, where $s \in [0,1]$. Is D bounded, linear? Prove your result. In addition, if D is bounded, find its norm $||D||$.
(b) Define $T : C[0,1] \to C[0,1]$ by $(Tx)(s) = \int_0^s x(t)dt$, where $s \in [0,1]$. Is T bounded, linear? Prove your result. In addition, if T is bounded, find its norm $||T||$.

Problem 3 (20 points.)
Let (e_k) be an orthonormal sequence in a Hilbert space H, and $x \in H$. Let $y = \sum_{k=1}^{n} < x, e_k > e_k$.
(a) Show that $||y||^2 = \sum_{k=1}^{n} | < x, e_k > |^2$.
(b) Show that $\sum_{k=1}^{\infty} < x, e_k > e_k$ converges.

Problem 4 (20 Points.)
Let H be a complex Hilbert space and let $T : H \to H$ be a bounded linear operator. Assume that $||T^*x|| = ||Tx||$ for all $x \in H$.
(a) Show that T is normal.
(b) Show that $||T^2|| = ||T||^2$.

Problem 5 (15 Points.)
Let X be a Banach space and (T_n) be a sequence of bounded linear operators on X. Assume that $(f(T_n x))$ is bounded for every $x \in X$ and every $f \in X'$, where X' is the dual space of X. Prove that $(||T_n||)$ is bounded.

Qualifying Exam: Ordinary Differential Equations I, April 2010

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 34 points each. To receive full credit, you need to justify all your statements.

1. a) Prove that the solution to the problem $x' = f(t,x)$ with $x(\tau) = \xi$ and $f(t,x)$ continuous, can be written as $\psi(t,\tau,\xi) = \xi + \int_\tau^t f(s,\psi(s,\tau,\xi))ds$
   b) Prove that solution to the above integral equation exist and is unique if $|f(t,x) - f(t,y)| < K|x-y|$ for some finite constant K and all (t,x), & (t,y) in the domain.

2. Consider the boundary value problem on $[0,1]$ for the differential equation $i\frac{dx}{dt} + \lambda x = 0$, with $x(1) = \beta x(0)$, where $i = \sqrt{-1}$ and β is a complex number.
   a) Find the eigenvalues
   b) Find the corresponding eigenfunctions
   c) Find a general condition on β that will make the eignevalues to be all real.

3. Find the solution to the system:

$$\begin{aligned} X_1' &= 3X_1 + 2X_2 + t \\ X_2' &= 2X_1 + 3X_2 - t \end{aligned}$$

   with the initial condition that $\mathbf{X}(0) = \xi$.

ABSTRACT ALGEBRA
(Dr. Mieczyslaw K. Dabkowski)
Qualifying Exam
April 7, 2010

Name____________________________

Instructions. *Please solve any* **five problems** *from the list of the following problems (show all your work).*

1. Let G be an abelian group. For $g \in G$, let $\mid g \mid$ denotes the order of g. Prove that $\{g \in G \mid \mid g \mid < \infty\}$ is a subgroup of G.

2. Let G be a group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

3. Prove that a group G of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

4. Let $\varphi : G \to H$ be an epimorphism of groups. Show that if H is not abelian then G is not abelian. Use this result to show that the group given by the following presentation:
$$G = \langle x, y \mid x^2 = 1,\ xy = y^{-1}x \rangle$$
is not abelian. *Hint*: Consider the epimorphism $\psi : G \longrightarrow D_6$, where
$$D_6 = \langle r,\ s \mid r^3 = 1,\ s^2 = 1,\ sr = r^{-1}s \rangle.$$
is dihedral group of 6 elements.

5. Determine whether the following polynomials are irreducible in the rings indicated. For these that are reducible, determine their factorization into irreducibles.
   - $x^2 + x + 1$ in $\mathbb{Z}_2[x]$.
   - $x^4 + 1$ in $\mathbb{Z}_5[x]$.
   - $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$.

6. Show that an ideal $M \neq R$ in a commutative ring R with identity 1_R is maximal if and only if for every $r \in R \backslash M$, there is $x \in R$ such that $1_R - rx \in M$.

   Hint: Consider the ideal $I = (r) \subset R$, and define $J = I + M$. Show that $R = J$ and conclude that $1_R = a + m$ where, $a \in I = (r)$.

7. Let D_4 be dihedral group of 4 elements[1] (the group of symmetries of a segment) and D_{16} be dihedral group of 16 elements (group of symmetries of regular octagon). If we regard D_4 as a subgroup of D_{16} ($D_4 \leq D_{16}$), find
$$N(D_4)/D_4,$$
where $N(D_4)$ denotes the normalizer of D_4 in D_{16}.

[1] Recall that the dihedral group of $2n$ elements (group of symmetries of regular n-gon) has the following presentation:
$$D_{2n} = \langle r,\ s \mid r^n = 1,\ s^2 = 1,\ sr = r^{-1}s \rangle$$

Complex Analysis Qualifying Exam
(2010; by J. Turi)

1) The function $f(z) = \frac{1}{z}$ is holomorphic on the set

$$U = \{z \in C : 1 < |z| < 2\}.$$

Prove that f does not have a holomorphic antiderivative on U.

2) Compute the complex line integral:

$\int_\gamma (\bar{z} + z^2\bar{z})dz$, where γ is the unit square (center at (0.0)) with clockwise orientation .

3) Suppose that f and g are entire functions and that g never vanishes. If $|f(z)| \leq |g(z)|$ for all z, then prove that there is a constant C such that $f(z) = Cg(z)$. What if g does have zeros?

4) Let f be holomorphic on $U - \{P\}, P \in U, U$ open. If f has an essential singularity at P, then what type of singularity does $\frac{1}{f}$ have at P? What about when f has a removable singularity or a pole at P?.

5) Estimate the number of zeros of the function $f(z) = z^{10} + 10z + 9$ in $U = D(0,1)$.

Choice exam

Name: ______________________________

Qualifying Exam, April 2010
Math Methods in Medicine and Biology

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (25 points.) Suppose that every pair of rabbits can produce only twice, when they are one and two months old, and that each time they produce exactly one new pair of rabbits. Assume that all rabbits survive. Let

$$\begin{aligned} R_n^0 &= \text{number of newborn pairs in generation } n, \\ R_n^1 &= \text{number of one-month-old pairs in generation } n, \\ R_n^2 &= \text{number of two-month-old pairs in generation } n. \end{aligned}$$

Assume that it is started with a single pair of newborn rabbits in the first generation, i.e. $R_0^0 = 1$ and $R_1^0 = 1$.

(a) Show that R_n^0 satisfies the equation $R_{n+1}^0 = R_n^0 + R_{n-1}^0$.

(b) Find the numbers of pairs of newborn, one-month-old, and two-month-old rabbits after n generations, i.e., find R_n^0, R_n^1 and R_n^2.

Problem 2 (25 points.) Consider the following nonlinear difference equation for density-limited population growth:

$$N_{t+1} = \lambda N_t (1 + aN_t)^{-b}, \quad \lambda > 0, a > 0, b > 0.$$

(a) Find all the steady states.

(b) Decide the stability conditions for each steady state.

Problem 3 (30 points.) Consider the following model for bacterial growth in a chemostat.

$$\begin{aligned} \frac{dN}{dt} &= \left(\frac{K_{max}C}{K_n + C}\right) N - \frac{FN}{V} \\ \frac{dC}{dt} &= \alpha \left(\frac{K_{max}C}{K_n + C}\right) N - \frac{FC}{V} - \frac{FC_0}{V}. \end{aligned}$$

Where $N(t)$ represents bacterial population density, $C(t)$ represents nutrient concentration in growth chamber, and K_{max}, K_n, F, V, α, C_0 are positive constants.

a) Show that the equations can be written in the following dimensionless form:

$$\begin{aligned} \frac{dN}{dt} &= \alpha_1 \left(\frac{C}{1 + C}\right) N - N \\ \frac{dC}{dt} &= -\left(\frac{C}{1 + C}\right) N - C + \alpha_2. \end{aligned}$$

Determine α_1 and α_2 in terms of the original parameters.

b) Find the positive steady state and its stability property.

Problem 4 (20 points.) Consider the following system of ordinary differential equations,

$$\begin{aligned} \frac{dx}{dt} &= x - \frac{x^3}{3} + y, \\ \frac{dy}{dt} &= -x. \end{aligned}$$

(a) Find the steady state and its stability property.

(b) Show that there is a limit cycle trajectory.

MATH 6320 - CHOICE EXAMINATION

[redacted]

QUALIFYING EXAMINATIONS - SPRING 2010

APRIL 9th, 2010 - CLOSED BOOK

To be completed between 9am and Noon

V. Ramakrishna

- **I** Consider an element of $SO(3, R)$ represented by a unit quaternion u. Find an expression for its axis of symmetry. Justify your work.

  (5 points)

- **II** i) Define the product of two octonions, each represented as a pair of quaternions; ii) State the matrices, ω_a, θ_a which represent left multiplication and right multiplication by an octonion $a \in \mathcal{O}$ (all auxilliary matrices in your definition must be properly defined); iii) Why is $\omega_{ab} \neq \omega_a\omega_b$?; iv) Use the identity $(ab)xa = a(bx)a$ to show that the matrices in iii) are nevertheless similar; v) Show explicitly that $det(\omega_a) = \mid a \mid^8$; vi) Use iv) and v) to show that the octonion norm is multiplicative $(2 + 3 + 2 + 3 + 3 + 2 = 15$ points)

- **III)** Let A and B be positive definite matrices satisfying $A \geq B$. Show $B^{-1} \geq A^{-1}$. Hint: Consider the Schur complements of A and B^{-1} in $\begin{pmatrix} A & I \\ I & B^{-1} \end{pmatrix}$.

  (8 points)

- **IV** i) Show $A = (min\{i, j\})$ is positive semidefinite; ii) Show, using i) and the fact that congruence preserves positive semidefinitenss, that $B = (\frac{1}{max\{i,j\}})$ is also positive semidefinite; iii) Let C be defined by $c_{ij} = c_{ji}$ and $c_{ij} = \frac{i}{j}$ for $i \leq j$. Show C is positive semidefinite.

(3 + 2 + 2 = 7 points)