Name: ______________________________

Qualifying Exam, April 2009
Real Analysis I

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Solve 4 of the following 5 problems. You must clearly indicate which 4 are to be graded.

Problem 1 (25 points.)
If μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then for any $E \subset X$,

$$\mu^*\left(E \cap (\cup_{j=1}^\infty A_j)\right) = \sum_{j=1}^\infty \mu^*(E \cap A_j).$$

Problem 2 (25 points.)
Let $C \subset [0,1]$ be the Cantor set. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \notin C; \\ 0 & \text{if } x \in C. \end{cases}$$

(a) Is f Lebesgue measurable on $\mathbb{R}$? Justify your answer.
(b) Is f Riemann integrable on $[0,1]$? Is f Lebesgue integrable on $[0,1]$? Justify your answer.

Problem 3 (25 points.)
Compute the following limit and justify the calculations. (Hint: Use the properties of the function $\frac{\sin y}{y}$.)

$$\lim_{n\to\infty} \int_0^\pi n \sin\left(\frac{x}{n}\right) dx$$

Problem 4 (25 points.)
Let $\{f_n\}$ be a sequence of real-valued functions on $\mathbb{R}$. Let m be the Lebesgue measure. Show that if $f_n \to f$ in $L^1(\mathbb{R}, m)$, then $f_n \to f$ in measure. Is the converse true? Justify your answer.

Problem 5 (25 points.)
Let f and g be real-valued absolutely continuous functions on $[a,b]$, $a < b$.
(a) Show that the product fg is also absolutely continuous on $[a,b]$. (Hint: first show that f and g are bounded.)
(b) Show that $\int_a^b [f'(x)g(x) + f(x)g'(x)]dx = f(b)g(b) - f(a)g(a)$.

Name: ______________________

Qualifying Exam, April 2009
Real Analysis II

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (25 points.)
Let X abd Y be vector spaces. Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $\mathcal{R}(T) = Y$ iff T^{-1} exists. Here $\mathcal{R}$ denotes the range of T.

Problem 2 (25 points.)
Let $C[0,1]$ denote the normed space of all continuous real-valued functions on $[0,1]$, with the norm defined by $\|x\| = \max_{s\in[0,1]} |x(s)|$. On $C[0,1]$, define an operator T by $Tx = x(s)y_0(s)$, where $y_0 \in C[0,1]$ is fixed. Show that T is a bounded linear operator and find its norm $\|T\|$.

Problem 3 (25 points.)
Let H be a Hilbert space.
(a) Prove that if $x_n \to x$ and $y_n \to y$, then $< x_n, y_n > \to < x, y >$.
(b) Let $M \subset H$, $M \neq \emptyset$. Prove that $M^{\perp} \subset (\overline{\text{span} M})^{\perp}$.

Problem 4 (25 points.)
(a) Let X be a normed space and let $x_0 \neq 0$ be any element of X. Use the Hahn-Banach Theorem for normed spaces to show that there exists a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = 1$ and $\tilde{f}(x_0) = \|x_0\|$.
(b) If X in (a) is a Hilbert space, find $\tilde{f}$ in this case.

Ordinary Differential Equations
Qualifying Exam
April, 2009

1. Prove that if $f \epsilon C(D)$ and if f satisfies a Lipshitz condition in D with Lipschitz constant L, then the initial value problem $x' = f(t, x)$ and $x(\tau) = \xi$, with $(\tau, \xi) \epsilon D$ has at most one solution on any interval $|t - \tau| \leq d$. Note: D is an open, connected, nonempty subset of R^2 and $(t, x) \in D$.

2. Prove that if $\Phi(t)$ is a fundamental set of solutions of $x' = A(t)x$ with periodic coefficient $A(t) = A(t+T)$, then $\Phi(t+T)$ is also a fundamental set of solutions of $x' = A(t)x$. Furthermore exist a non-singular periodic matrix $P(t)$, with period T and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$.

3. Suppose that for a continuous function $f(t)$ we are given that the equation

$$x' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix} x + f(t)$$

has at least one solution $\phi_p(t)$ which satisfies

$$sup\{|\phi(t)| : \tau \leq t < \infty\} < \infty.$$

Show that all the solutions of above ODE satisfy this boundedness condition.

4. For what values of a and b, with $0 \leq a < b \leq \pi$, is the differential operator L defined by

$$Ly = \frac{d}{dt}[(2 + \sin t)\frac{dy}{dt}] + (\cos t)y, \qquad y(a) = y(b), \quad \text{and} \quad y'(a) = y'(b),$$

self-adjoint?

ABSTRACT ALGEBRA

(DR. Mieczyslaw K. Dabkowski)

Qualifying Exam

April 8, 2009

Name______________________

Instructions. *Please solve any five problems from the list of the following problems (show all your work).*

1. Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.
2. Prove that if $|G| = pq$ for some primes p and q (not necessarily distinct) then either G is abelian or $Z(G) = \{1\}$.
3. Prove that if $|G| = 132$ then G is not simple.
4. Let G be a group given by the following presentation

$$G = \langle x, y \mid x^2 = y^2,\ x^4 = 1,\ xyx^{-1} = y^{-1} \rangle$$

Show that G is a finite group which is isomorphic to Q_8 (group of quaternions).

5. Let $f(x)$ be a polynomial in $F[x]$, where F is a field. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.
6. Recall, an element $x \in R$ is called nilpotent if $x^m = 0$ for some $m \in \mathbb{Z}_+$. Suppose that x be an element of the commutative ring R with unity.
   (a) Prove that x is either zero or a zero divisor.
   (b) Prove that rx is nilpotent for all $r \in R$.
   (c) Prove that $1 + x$ is a unit in R.

COMPLEX ANALYSIS - APRIL 2009 QUALIFYING EXAMINATION

TOBIAS HAGGE

Show work and justifications clearly.
$B_r(z)$ denotes the open ball of radius r centered at z.
A *region* is a connected open subset of $\mathbf{C}$.

(1) Prove that the ratio test works, i.e. if $lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|} = R$ then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R.

(2) Let f be analytic on a region Ω, with $f(\Omega) \subset B_1(1)$. Prove that $\int_\gamma \frac{f'(z)dz}{f(z)} = 0$ for all closed curves $\gamma \subset \Omega$.

(3) State and prove Liouville's theorem.

(4) Compute $lim_{k \to \infty} \int_{-k}^{k} \frac{dx}{x-i}$ using complex methods. Does $\int_{-\infty}^{\infty} \frac{dx}{x-i}$ exist?

(5) Prove the following generalization of Schwarz' lemma, assuming the original lemma, which is the $n = 1$ case. Let $f : B_1(0) \to B_1(0)$ be analytic. If $f^{(m)}(0) = 0$ for all $m < n$, then $|f^{(n)}(0)| \leq n!$, and $\frac{|f(z)|}{|z|^n} \leq 1$. If $|f^{(n)}(0)| = n!$ or $\frac{|f(z)|}{|z|^n} = 1$ for some $z \in B_1(0)\backslash\{0\}$, then $f(z) = cz^n$. Hint: induct and use Schwarz' lemma.

Student's choice '09

MATH 6319 - CHOICE EXAMINATION

[redacted]

QUALIFYING EXAMINATIONS - SPRING 2009

APRIL 10th, 2009 - CLOSED BOOK

To be completed between 9am and Noon

V. Ramakrishna

- **I** Derive the Fourier transform of a Gaussian, by obtaining a differential equation for it (No credit for any other method).

  (8 points)

- **II)** Let G_B be the automorphism group of a non-degenerate bilinear form. Let $A \in G_B$. Show that the eigenvalues of A arise in reciprocal pairs.

  (8 points)

- **III** Let $A = \Sigma_{i=1}^{r} \sigma_i u_i v_i^*$ be the singular value decomposition of A.

  - i) What are the σ_i?
  - ii) The u_i are orthonormal eigenvectors of a matrix related to A? Which matrix is that and what are the corresponding eigenvalues?
  - iii) Same Q as ii), but this time for the v_i.
  - iv) Suppose you have already found the u_i. How can the v_i be then found **without** having to solve an eigenvalue problem? Verify your claim.

  $(1 + 1 + 1 + 6 = 9$ points)

Student's choice '09

Numerical Analysis Qualifying Exam
(2009; by J. Turi)

1) Show that if U is a finite-dimensional subspace of a normed space X, then for every element in X there exists a best approximation with respect to U.

2) Discuss the application of projection methods for the solution two-point boundary value problems.

3) Show that if A is a diagonally dominant matrix, then both the Jacobi-method and the Gauss-Seidel- method converge.

4) State and prove a result for the convergence of the Newton method for the solution of systems of nonlinear equations.

5) Analyze the convergence of a semi-discrete finite difference scheme when applied to the heat equation.