Name: ______________________________

Qualifying Exam, April 2008
Real Analysis I

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (20 points.) Prove or disprove (by a counterexample) the following statements:
a. A countable subset of $\mathbb{R}$ has Lebesgue measure zero.
b. If a subset of $\mathbb{R}$ has Lebesgue measure zero then it is countable.

Problem 2 (20 points.)
Let $(X, \mathcal{M}, \mu)$ be a measure space.
a. Prove that μ is continuous from below, that is, if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu(\cup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \mu(E_j)$.

b. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in X and let $E = \cup_{k=1}^{\infty} \cap_{j=k}^{\infty} E_j$. Prove that $\mu(E) \leq \liminf \mu(E_j)$.

Problem 3 (20 points.) Let f be a real-valued function on $\mathbb{R}$. Which of the following statements are true? Justify your answers.
(i) If f is measurable, then $|f|$ is measurable.
(ii) If $|f|$ is measurable, then f is measurable.

Problem 4 (20 points.)
Let (f_n) be a sequence of integrable functions on $[0,1]$ such that $0 \leq f_{n+1} \leq f_n$ for all n and $f = \lim_{n\to\infty} f_n$. Show that $f = 0$ a.e. iff $\lim_{n\to\infty} \int f_n = 0$.

Problem 5 (20 points.) Compute the following limit and justify the calculations. (Hint: Use the dominated convergence theorem.)

$$\lim_{n\to\infty} \int_0^{\infty} \frac{2n^2 + \sin(n^2x^2+1)}{n^2+x^2} e^{-x} dx$$

Real Analysis II. Qualifying Exam 2008

1) Show that the subspace $Y \subset C[a, b]$ consisting of all continuous functions $x \in C[a, b]$ such that $x(a) = x(b)$ is complete.

2) Let $T : D(T) \to Y$ be a linear operator whose inverse exists. If $\{x_1, x_2, ..., x_n\}$ is a linearly independent set in $D(T)$, show that the set $\{Tx_1, Tx_2, ..., Tx_n\}$ is linearly independent.

3) Show that in a Hilbert space H, convergence of $\sum \|x_i\|$ implies the convergence of $\sum x_i$..

4) Show that every Hilbert space H is reflexive.

5) State the Open Mapping Theorem. Show that an open mapping need not map closed sets onto closed sets.

Qualifying Exam: Ordinary Differential Equations I, April. 2008

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 20 points each. To receive full credit, you need to justify all your statements.

1. Consider the initial value problem $x' = x^{1/5}$ and $x(0) = x_0$ for $t \geq 0$.

   a) Is the solution to the above IVP unique if $x_0 = 1$? Clearly justify/prove your answer.

   b) Is the solution to the above IVP unique if $x_0 = 0$? Clearly justify/prove your answer.

2. Consider the differential operator L, defined by the operation $Lu = i\frac{du}{dt}$, acting on differentiable functions u define over $[0,1]$ with the condition that $u(1) = \alpha u(0)$, where $i = \sqrt{-1}$ and α is a complex number. Find a general condition on α that will make the differential operator L to be self adjoint.

3. Consider the boundary value problem on $[0, \pi]$ for the equation

$$x'' + \lambda x = 0,$$

   with $x'(0) = 0$ and $x(\pi) = 0$.

   a. Are there any eigenvalues which are not real?

   b. Find the eigenvalues

   c. Find the corresponding eigenfunctions.

4. a) Find the Jordan form of the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

   b) Find e^{At}.

ABSTRACT ALGEBRA
(DR. Mieczyslaw K. Dabkowski)
Qualifying Exam
April 9, 2008

Name________________________

Instructions. *Please solve any five problems from the list of the following problems (show all your work).*

1. Prove that if $|G| = p^2$ for some prime p then G is an abelian group.
2. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.
3. Let G be a group given by the following presentation
$$G = \langle x, y \mid y^2 = 1,\ xyx^{-1} = y^{-1} \rangle$$
   (a) Show that G is infinite.
   (b) Show that G is not abelian group.
4. Show that the group $G = \langle x,\ y \mid x^2y^{-2} = 1,\ x^2(xy)^{-n} = 1,\ x^2 = 1 \rangle$ is isomorphic to D_{2n}.
5. Let R be a commutative ring with unity 1_R and I, J and M be ideals in R.
   (a) Show that an ideal $M \neq R$ is maximal if and only if for every $r \in R \backslash M$, there is $x \in R$ such that $1_R - rx \in M$.
   (b) Prove that IJ is an ideal contained in $I \cap J$.
   (c) Give an example where $IJ \neq I \cap J$.
   (d) Show that if $I + J = R$ then $IJ = I \cap J$.
6. Show that the ideal $I = (2, x) \subset \mathbb{Z}[x]$ is not a principal ideal of $\mathbb{Z}[x]$. Is I maximal ideal in $\mathbb{Z}[x]$?

COMPLEX VARIABLES
Qualifying Examination, April 11th, 2008
SPRING 2008
V. Ramakrishna

- **Q 1** i) Define what it means for a map $F : C \to C$ to be C-linear; ii) State two equivalent conditions for C-linearity; iii) Show that an angle-preserving real-differentiable map on a connected domain is either holomorphic or anti-holomorphic (you may assume the characterization of R-linear angle-preserving maps from C to C).

  $(3 + 4 + 8 = 15$ points)

- **Q2** i) State Schwarz's lemma; ii) Use it show that an automorphism of the unit disc which fixes the origin is a rotation; iii) Assuming the form of the linear fractional transformations in $\mathrm{A}ut(E)$, characterize (i.e., state and prove) $\mathrm{A}ut(E)$.

  $(4 + 5 + 11 = 20$ points)

- **Q 3**

  i) State the open-mapping theorem and use it show that a holomorphic function, whose absolute value is constant, must itself be constant. (you may assume, but should state, the existence lemma for zeroes usually used in its proof) ii) Define the winding number of a closed path. iii) State the Generalized Cauchy theorem (your statement must be of the form of three equivalent conditions on a closed path in an open set D).

$(5 + 5 + 5 = 15 \text{ points})$

Student's choice - 2008

Name: ______________________________

Qualifying Exam, April 2008
Math Methods in Medicine and Biology

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problem 1 (25 points.) In the circulatory system, red blood cells (RBCs) are constantly being destroyed in the spleen and created in the bone marrow. Assume that the spleen destroys a certain fraction of the RBCs daily and that the bone marrow produces a number proportional to the number lost on the previous day. Treat days as discrete time units, and let R_n be the number of RBCs in circulation on day n, and M_n be the number of RBCs produced by marrow on day n. Let f be the fraction of RBCs removed by the spleen every day, and γ be the number produced on day n for each cell lost on day $n-1$. Assume $0 < f < 1$ and $\gamma > 0$. Then the equations for R_n and M_n are

$$\begin{aligned} R_{n+1} &= (1-f)R_n + M_n \\ M_{n+1} &= \gamma f R_n \end{aligned}$$

a) Solve this system to find the general solution of R_n.
b) Find the condition for R_n remaining constant at large times in terms of the constants γ and f.

Problem 2 (25 points.) Consider the following nonlinear difference equation for population growth:

$$x_{n+1} = \frac{kx_n}{b + x_n}, \quad b > 0, k > 0.$$

(a) Find all the steady states.
(b) Decide the stability conditions for each steady state.

Problem 3 (30 points.) Consider the following predator-prey model

$$\begin{aligned} \frac{dN}{dt} &= rN - \frac{r}{K}N^2 - bNP, \\ \frac{dP}{dt} &= -cP + dNP. \end{aligned}$$

Where $N(t)$ and $P(t)$ represent prey and predator populations respectively, and r, K, b, c, d are positive constants.

a) Show that the equations can be written in the following dimensionless form:

$$\begin{aligned} \frac{dN}{dt} &= N - N^2 - NP, \\ \frac{dP}{dt} &= -\rho P + aNP. \end{aligned}$$

Determine ρ and a in terms of the original parameters.

b) Find the positive steady state and its stability properties.

Problem 4 (20 points.) Consider the two species competition model

$$\begin{aligned}\frac{dN_1}{dt} &= rN_1\frac{K_1 - N_1 - b_{12}N_2}{K_1},\\ \frac{dN_2}{dt} &= rN_2\frac{K_2 - N_2 - b_{21}N_1}{K_2}.\end{aligned}$$

where N_1 and N_2 are the population densities of species 1 and 2, r_1, r_2, K_1, K_2, b_{12}, b_{21} are positive constants. Show that there is no limit cycles exit.

Student's choice/2008

PRINCIPLES & TECHNIQUES OF APPLIED MATHEMATICS
2008 QUALIFYING EXAMINATION

Q1 Define the adjoint of a linear map with respect to a non-degenerate bilinear or sesquilinear map. Characterize self-adjoint, anti-self adjoint and form-preserving maps when the form is bilinear with coefficient matrix J_{2n}.

(12 points)

Q2 Calculate the Fourier transform of a Gaussian.

(12 points)

Q3 Let A, B be positive matrices. Show that the Kroecker product and Schur- Hadamard product of A and B are also positive.

(14 points)

Q4 State the rank-nullity theorem. Use it show that the problem of finding a degree k polynomial to interpolate the data points $(x_i, y_i), i = 1, \ldots, k+1$, with the x_i distinct admits a unique solution.

(12 points)