

# Real Analysis I. Qualifying Exam 2007

- 1) Show that every open set of real numbers is measurable.
- 2) Show that if  $f$  is a measurable function and  $f = g$  almost everywhere, then  $g$  is measurable.
- 3) Let  $f$  be a nonnegative integrable function. Show that

$$F(x) = \int_{-\infty}^x f$$

is continuous (by using the Monotone Convergence Theorem).

- 4) Argue that if  $f$  is absolutely continuous, then  $f$  has a derivative almost everywhere.
- 5) Let  $g$  be an integrable function on  $[0, 1]$ . Show that there is a bounded measurable function  $f$  such that  $\|f\| \neq 0$  and

$$\int fg = \|g\|_1 \|f\|_\infty.$$

## Real Analysis II. Qualifying Exam 2007

- 1) Let  $M \subset l^\infty$  be the subspace consisting of all sequences  $x = (\xi_i)$  with at most finitely many nonzero terms. Find a Cauchy sequence in  $M$  which does not converge in  $M$ , so that  $M$  is not complete. Why does that not contradict the completeness of  $l^\infty$ ?
- 2) Show that a compact subset  $M$  of a metric space is closed and bounded. Give an example showing that the converse is in general false.
- 3) Show that if  $Y$  is a finite dimensional subspace of a Hilbert space  $H$ , then  $Y$  is complete.
- 4) State the Hahn-Banach Theorem for normed spaces and assuming it show the existence of a bounded linear functional  $f$  on the normed space  $X$  such that
$$\|f\| = 1, \text{ and } f(x_0) = \|x_0\|, \text{ where } x_0 \neq 0 \text{ and } x_0 \in X.$$
- 5) Let  $T_n \in B(X, Y)$ , where  $X$  is a Banach space. If  $(T_n)$  is strongly operator convergent, show that  $(\|T_n\|)$  is bounded.

Qualifying Exam: Ordinary Differential Equations I, April 2007

**THIS IS A CLOSED BOOK, CLOSED NOTES EXAM**

Problems count 20 points each. To receive full credit, you need to justify all your statements.

1. For  $x \in \mathbf{R}$ , consider the IVP  $\frac{d}{dt}x = -x + g(t)$ ,  $x(0) = \xi$ , where  $g(t)$  is continuous and  $|g(t)| \leq 1$  for all  $t$ . Show that the IVP has a unique and bounded solution for all  $t \geq 0$ .
2. Consider the initial value problem  $\frac{d}{dt}x = x^{1/3}$  and  $x(0) = 0$  for  $t \geq 0$ . Show that the problem has infinitely many different solutions.
3. Consider the boundary value problem on  $[0, 1]$  for the equation

$$i \frac{dx}{dt} + \lambda x = 0,$$

with  $x(1) = \alpha x(0)$ , where  $i = \sqrt{-1}$  and  $\alpha$  is a complex number.

- a. Find the eigenvalues
  - b. Find the corresponding eigenfunctions.
  - c. Find a general condition on  $\alpha$  that will make the eigenvalues to be all real.
4. Consider the differential operator  $L = i \frac{d}{dt}$  acting on differentiable functions  $\phi(t)$  on  $[0, 1]$  with  $\phi(1) = \alpha \phi(0)$ . Note that  $\alpha$  is a complex number and  $i = \sqrt{-1}$ .
    - a. Find a general condition on  $\alpha$  that will make the differential operator  $L$  to be self-adjoint.
    - b. Show that the eigenfunctions of this self-adjoint operator form an orthogonal set.
  5. Find the fundamental set of solutions to the system  $\frac{d}{dt}x = Ax$ , with:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

**ABSTRACT ALGEBRA**  
(DR. Mieczysław K. Dabkowski)  
**Qualifying Exam**  
April 11, 2007

Name \_\_\_\_\_

**Instructions.** Please solve any four problems from the list of the following problems (show all your work).

1. Prove that if  $|G| = pq$  for some primes  $p$  and  $q$  (not necessarily distinct) then either  $G$  is abelian or  $Z(G) = \{e\}$ .
2. Let  $G$  be a group given by the following presentation

$$G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$$

- (a) Show that  $G$  is infinite.
  - (b) Show that  $G$  is not abelian group.
3. Show that:
    - (a) Dihedral group  $D_8 = \langle r, s \mid r^4 = 1, s^2 = 1, rs = sr^{-1} \rangle$  and the quaternion group  $Q_8 = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle$  are not isomorphic.
    - (b) The group  $\mathbb{Z} \oplus \mathbb{Z}$  is not cyclic.
  4. Let  $R$  be a commutative ring with unity  $1_R$  and  $I, J$  be ideals in  $R$ .
    - (a) Prove that  $I + J$  is the smallest ideal in  $R$  which contains both  $I$  and  $J$ .
    - (b) Prove that  $IJ$  is an ideal contained in  $I \cap J$ .
    - (c) Give an example where  $IJ \neq I \cap J$ .
    - (d) Show that if  $I + J = R$  then  $IJ = I \cap J$ .
  5. Show that the ring of polynomials  $\mathbb{Z}[x]$  is not a principal ideal domain.

Qualifying Examination in Complex Analysis  
*Closed Book*

April 13, 2007

Write your name on each page

**Answer as many questions as you can.**

By using the definition of the derivative of the complex function argue that the function  $f(z) = \bar{z}$  is nowhere differentiable.

Assume that the function

$$w = f(z) = u(x, y) + i v(x, y)$$

is differentiable in a region  $G$ . Derive the Cauchy-Riemann equations.

Consider Cauchy's fundamental theorem:

If  $f(z)$  is differentiable in a simply connected region  $G$ , then

$$\int_C f(z) dz = 0,$$

where  $C$  is an arbitrary closed path lying in  $G$ .

E. Goursat proved the Cauchy theorem for a rectangle contained in  $G$  without the assumption of continuity of the derivative. The great significance of Goursat's theorem warrants to call the fundamental theorem Cauchy-Goursat theorem. Could you reconstruct Goursat's proof, or the same proof for a triangle in  $G$  as given in K. Knop's book: Theory of Functions.

We now come to the most important consequence of the Cauchy-Goursat theorem: State and prove the Cauchy integral formula for  $f(z)$ , which is differentiable throughout the region  $G$ .

Give the integral formula for  $f^{(n)}(z)$  as well.

Find the Laurent expansion for

$$f(z) = \frac{1}{(2z-1)(z-1)}$$

valid in

$$\frac{1}{2} < |z| < 1.$$

State the residue theorem.

Using the residue theorem compute the following integrals

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + x + 2} dx = \frac{2\pi}{\sqrt{7}}$$

Hint: The function  $\frac{1}{z^2+z+2}$  has a pole on the upper half plane. Compute the residue with the help of l'Hopital's rule.

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{3}.$$

Hint: The function  $\frac{z^2}{z^6+1}$  has three poles, the three sixth roots of  $-1$ , on the upper half plane. Compute the residues using l'Hopital's rule.

Illustrate the Weierstrass factor theorem by writing out the product representation of

$$\sin \pi z$$

(Remember that  $\sin \pi z$  has simple zeros at every integer)

Illustrate the Mittag-Leffler partial-fractions theorem by writing out

$$\pi \cot \pi z$$

in terms of its poles. Remember:  $\pi \cot \pi z$  has its poles of order unity and residue  $+1$  at every integer, hence with principal parts

$$\frac{1}{z-n} \quad \text{and} \quad \frac{1}{z+n},$$

respectively.

**Have a good time.**

TOPOLOGY — "Choice"  
(DR. Mieczyslaw K. Dabkowski)  
Qualifying Exam  
April 13, 2007

Name \_\_\_\_\_

**Instructions.** Please solve any 4 problems from the list of the following problems (show all your work).

1. Let  $\tau = \{U \subset \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .
  - (a) Show that  $\tau$  is a topology on  $\mathbb{R}$ .
  - (b) Show that topological space  $(\mathbb{R}, \tau)$  is not a Hausdorff space.
  - (c) Find  $\overline{A}$ , where  $A = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ . Justify your answer.
2. Consider the following topologies on  $\mathbb{R}$ .
  - $\tau_1 = \tau(\mathcal{B}_1)$ , where  $\tau(\mathcal{B}_1)$  denotes the topology generated by the basis  $\mathcal{B}_1 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$  (standard topology on  $\mathbb{R}$ )
  - $\tau_2 = \tau(\mathcal{B}_2)$ , where  $\tau(\mathcal{B}_2)$  denotes the topology generated by the basis  $\mathcal{B}_2 = \{(a, b) \mid a < b; a, b \in \mathbb{R}\}$  (lower limit topology on  $\mathbb{R}$ )

Show that:

- (a)  $\tau_1 \subseteq \tau_2$ ;
  - (b)  $\tau_1 \neq \tau_2$ ;
3. Let  $G$  be an open subset of  $X$  and  $A \subset X$ .

- (a) Prove that  $\overline{G \cap A} = \overline{G} \cap \overline{A}$ .

*Hint:* In order to show that

$$\overline{G \cap A} \subseteq \overline{G} \cap \overline{A}$$

you may start by showing that for any  $A \subset X$  and any open subset  $G \subset X$  if  $G \cap \overline{A} \neq \emptyset$ , then  $G \cap A \neq \emptyset$ .

- (b) Show by example that the condition for  $G$  to be open is necessary.

4. Let  $X$  be a set and  $d : X \times X \rightarrow \mathbb{R}$  be a metric on  $X$ . Show that the  $\rho : X \times X \rightarrow \mathbb{R}$  defined as follows:

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a metric on  $X$ .

*Hint:* Consider function  $f : (0, \infty) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{x}{1+x}$  and show that  $f$  is increasing.

5. Let  $f : S^1 \rightarrow \mathbb{R}$  be a continuous function, where  $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ . Show that there exists a point  $\mathbf{x} \in S^1$  such that  $f(\mathbf{x}) = f(-\mathbf{x})$ .

*Hint:* Consider function  $g : S^1 \rightarrow \mathbb{R}$  defined by  $g(\mathbf{y}) = f(\mathbf{y}) - f(-\mathbf{y})$  and find  $g(\mathbf{y})$  and  $g(-\mathbf{y})$ .



**TOPOLOGY & GEOMETRY** — "Choice"  
 (DR. Mieczyslaw K. Dabkowski)  
**Qualifying Exam**  
 April 13, 2007

Name \_\_\_\_\_

**Instructions.** Please solve any 4 problems from the list of the following problems (show all your work).

1. Show that  $N = \{(x, y) \in \mathbb{R}^2 \mid (x + 1)^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 = 1\} \subset \mathbb{R}^2$ , where  $N$  is given the subspace topology induced by the standard topology on  $\mathbb{R}^2$  is not a one dimensional submanifold of  $\mathbb{R}^2$ .
2. Let  $M^2 = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$ , where  $F : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a differentiable function, such that  $\nabla F \neq 0$ . Show that:
  - $(\forall p \in M^2)[\nabla F \perp T_p M^2]$ ;
  - Find the equation of  $T_p M^2$ .
3. Let  $M$  be a differential manifold, and let  $X$  be a vector field on  $M$ . Find the flow generated by  $X$ , where
  - (a)  $M = \mathbb{R}^2$  and  $X(x, y) = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ .
  - (b)  $M = \mathbb{R}$  and  $X(x) = x^2 \frac{d}{dx}$ .
4. Let  $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$  be a 2-dimensional sphere in  $\mathbb{R}^3$ , where  $\mathbb{R}^3$  is given standard topology and  $S^2$  is given induced topology. Define  $p_N = (0, 0, 1)$  and  $p_S = (0, 0, -1)$ ,

$$\begin{aligned} \varphi &= (\varphi_1, \varphi_2) : S^2 \setminus \{p_N\} \rightarrow \mathbb{R}^2 \\ \varphi_i((x_1, x_2, x_3)) &= \frac{x_i}{1 - x_3}, \quad i = 1, 2; \end{aligned}$$

and

$$\begin{aligned} \psi &= (\psi_1, \psi_2) : S^2 \setminus \{p_S\} \rightarrow \mathbb{R}^2 \\ \psi_i((x_1, x_2, x_3)) &= \frac{x_i}{1 + x_3}, \quad i = 1, 2; \end{aligned}$$

Prove that  $S^2$  with the atlas  $\{(S^2 \setminus \{p_N\}, \varphi), (S^2 \setminus \{p_S\}, \psi)\}$  is a 2-dimensional differentiable manifold.

5. Let  $U$  is an open subset of  $\mathbb{R}^2$  and  $r : U \rightarrow M^2$  be a local parametrization of a regular surface  $M^2$ . Suppose that the matrix representing Riemannian metric  $\langle, \rangle$  on  $r(U) \subset M^2$  has the following form

$$\langle, \rangle = \begin{bmatrix} g(x, y) & 0 \\ 0 & 1 \end{bmatrix} = g(x, y) dx \otimes dx + dy \otimes dy, \quad \text{where } g : U \rightarrow \mathbb{R}_+ \text{ is a differentiable function.}$$

Find Christoffel symbols  $\Gamma_{ij}^k$  and system of differential equations describing geodesics on  $r(U)$ .

Qualifying Exam: Numerical Analysis, April 2007

"Choice"

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM

Problems count 20 points each. To receive full credit, you need to justify all your statements.

1. If  $A$  is a non-singular matrix and  $\|A - B\| * \|A^{-1}\| < 1$ , prove that matrix  $B$  is also non-singular.
2. A large class of iterative methods for solving  $Ax = f$  is based on splitting the matrix  $A = N - P$ , with  $\det|N| \neq 0$ , and using the iterative relation

$$N\mathbf{x}^{(\nu)} = P\mathbf{x}^{(\nu-1)} + \mathbf{f}, \quad \nu = 1, 2, \dots$$

Find a sufficient condition on the matrix  $M = N^{-1}P$  such that the above iterative method will converge if we start with some arbitrary vector  $\mathbf{x}^{(0)}$ . Please justify your statement.

3. Consider the problem of finding  $\alpha$ , solution to the fixed point problem  $\alpha = f(\alpha)$  for  $f(x) = \frac{20}{x^2 + 2x + 10}$ .
  - (a) Compute  $x_n$ , estimates for the fixed point, using the fixed point iterations method. Take  $x_0 = 1$  and find  $x_n$  for  $n = 1, 2$ .
  - (b) Show that  $|x_{n+1} - x_n| = |1 - f'(\xi)| * |\alpha - x_n|$ , for some  $\xi$  between  $\alpha$  and  $x_n$ .
4. Write a simple program, using trapezoidal integration, that will compute the integral  $\int_1^\infty \frac{e^{-x}}{x^{1/\pi}} dx$ . Don't calculate any numbers, just make sure everything is well defined and computable.
5. We are given the following rule  $y_{n+1} = y_{n-1} + 2h * f(x_n, y_n)$  to help us solve  $y' = f(x, y)$ , with  $y(x_0) = y_0$ . (a) Prove that this method satisfies consistency condition  $\tau(h) = O(h^m)$ , and find  $m$  for this method. (b) Is this method stable? Please justify your statement. (c) Write a simple program which uses the above method to solve  $y' = f(x, y)$ , with  $y(x_0) = y_0$ .

Please see the back for some equations and information which may or may not be helpful in solving above problems

Some equations and information which may or may not be helpful in solving the problems

- Taylor's Theorem:  $f(x) = f(a) + f'(a)(x - a) + \dots + f^{(n)}(a)(x - a)^n/n! + R_n(x)$
- $1 + x + \dots + x^n = (1 - x^{n+1})/(1 - x)$ , when  $x \neq 1$
- Mean Value Theorem: Exist  $c$  in  $(a, b)$  such that:  $f(b) - f(a) = f'(c)(b - a)$
- Fixed point problem:  $x = f(x)$ . Solution by iteration is:  $x_{n+1} = f(x_n)$ .
- For fixed point iteration to converge over  $[a, b]$ , need:  $\lambda = \max_{a \leq x \leq b} |f'(x)| < 1$ .
- Root of  $f(x) = 0$  by Newton's method is found by:  $x_{n+1} = x_n - f(x_n)/f'(x_n)$ .
- For Newton's method to converge:  $|\alpha - x_0| < 1/M$ , where  $M = \frac{\max |f''|}{2 \min |f'|}$ .
- $p$ , the order of convergence of a sequence of iterate to  $\alpha$  is defined as:  
 $|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p$ .
- Lagrange's formula to interpolate  $f(x)$  is  $L(x) = \sum_{j=0}^n f(x_j) * l_{j,n}(x)$ , where  $l_{j,n}(x) = \prod_{i \neq j}^n \frac{(x - x_i)}{(x_i - x_j)}$
- Finite Fourier Transform of order  $m$  is  $d_k = (1/m) \sum_{j=0}^{m-1} W_m^{jk} f_j$ , for  $k = 0, 1, \dots, (m-1)$ .  
 Where  $W_m = e^{-2\pi i/m}$ , and  $d_{k+m} = d_k$  for  $-\infty < k < \infty$ .
- Lipschitz Condition for  $f(x, y)$  continuous in  $D$  is:  $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$   
 for all  $(x, y_1)$  and  $(x, y_2)$  in  $D$ .
- Trapezoidal Quadrature =  $h[f_0/2 + f_1 \dots f_{n-1} + f_n/2]$ .
- Simpson's Quadrature =  $h[f_0 + 4f_1 + 2f_2 \dots + 2f_{n-2} + 4f_{n-1} + f_n]/3$ , with  $n$  even.
- Multistep Rule is  $y_{n+1} = \sum_{j=0}^p a_j y_{n-j} + h \sum_{j=-1}^p b_j f(x_{n-j}, y_{n-j})$  for  $n \geq p$ .  
 Consistency condition for  $\tau(h) = O(h^m)$  are:  $\sum_{j=0}^p a_j = 1$ ,  $-\sum_{j=0}^p j a_j + \sum_{j=-1}^p b_j = 1$ ,  
 and  $\sum_{j=0}^p (-j)^i a_j + i \sum_{j=-1}^p (-j)^{i-1} b_j = 1$ , for  $i = 1, 2, \dots, m$
- The root condition for multistep methods is that roots of the polynomial  
 $\rho(r) = r^{p+1} - \sum_{j=0}^p a_j r^{p-j}$ , be less than one in magnitude and simple if they are equal to one. That is  $|r_j| \leq 1$  and if  $|r_j| = 1$ , then  $\rho'(r_j) \neq 0$  for  $j=0, 1, 2, \dots, p$ .
- Euler's Method for IVP is  $y_{n+1} = y_n + hf(x_n, y_n)$
- Midpoint Method for IVP is  $y_{n+1} = y_{n-1} + 2hf(x_n, y_n)$
- RK4 Method for IVP is  $y_{n+1} = y_n + (h/6)[V_1 + 2V_2 + 2V_3 + V_4]$ , where  $V_1 = f(x_n, y_n)$ ,  
 $V_2 = f(x_n + h/2, y_n + hV_1/2)$ ,  $V_3 = f(x_n + h/2, y_n + hV_2/2)$ , and  $V_4 = f(x_n + h, y_n + hV_3)$ .

Name: \_\_\_\_\_

**Qualifying Exam, April 2007**  
**Math Methods in Medicine and Biology** "Choice"

**THIS IS A CLOSED BOOK, CLOSED NOTES EXAM**

**Problem 1** (20 points.)

Consider the Fibonacci Sequence,  $x_n = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \dots\}$ , show that  $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}$ .

**Problem 2** (30 points.)

Consider the following system of difference equations:

$$\begin{aligned} N_{t+1} &= rN_t \exp(-aP_t), \\ P_{t+1} &= N_t [1 - \exp(-aP_t)], \end{aligned}$$

where  $r > 1$  and  $a > 0$  are constants.

Find the positive steady state and show that this steady state can never be stable.

**Problem 3** (30 points.)

In this problem we examine a continuous plant-herbivore model. We shall define  $q$  as the chemical state of the plant and  $I$  as the herbivore density. Typical equations suggested for such system are

$$\begin{aligned} \frac{dq}{dt} &= K_1 - K_2 q I (I - I_0), \\ \frac{dI}{dt} &= K_3 I \left(1 - \frac{K_4 I}{q}\right). \end{aligned}$$

where  $I_0, K_1, K_2, K_3$  and  $K_4$  are positive constants.

a) Show that the equations can be written in the following dimensionless form:

$$\begin{aligned} \frac{dq}{dt} &= 1 - KqI(I - 1), \\ \frac{dI}{dt} &= \alpha I \left(1 - \frac{I}{q}\right). \end{aligned}$$

Determine  $K$  and  $\alpha$  in terms of original parameters.

b) Find the equations for the steady state and its stability properties.

**Problem 4** (20 points.)

Consider the following system of ordinary differential equations,

$$\begin{aligned} \frac{dx}{dt} &= x - \frac{x^3}{3} + y, \\ \frac{dy}{dt} &= -x. \end{aligned}$$

Show that there is a limit cycle trajectory.

"choice"

PRINCIPLES AND TECHNIQUES - 2007  
QUALIFYING EXAMINATION  
CLOSED BOOK

V. RAMAKRISHNA

• ALL WORK AND EXPLANATIONS MUST BE SHOWN CLEARLY

• Q1 Show that all the eigenevalues of a Hermitian matrix are real. Assuming that a Hermitian matrix can be diagonalized by a unitary matrix, show that every Hermitian matrix can be written as  $H = \sum_{i=1}^m \lambda_i v_i v_i^*$  for real numbers  $\lambda_i$ , length one vectors  $v_i$  which are orthogonal to each other.  $v_i^*$  is the transpose conjugate of the column vector  $v_i$ . (9 points)

• Q2 Let  $A$  be  $m \times n$ . In the factorization  $A = \sum_{i=1}^r \sigma_i u_i v_i^*$  what are the  $\sigma_i, u_i, v_i$ . How is  $r$  related to the rank of  $A$ . Assuming you have found the  $u_i$ , how may the  $v_i$  be found without solving a spectral problem?

(9 points)

• Q3 Use the Gram-Schmidt procedure to argue that the group  $O(n)$  of  $n \times n$  orthogonal matrices acts transitively on the sphere in  $R^n$ . What is the isotropy group at  $(1, 0, \dots, 0)$  of this action?

(9 points).

• Q4 Define i) a persymmetric matrix, and ii) a Hamiltonian matrix

(3 pts)