Qualifying Exam
Math 6301 August 2020
Real Analysis

QE ID______________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Let $A \subset \mathbb{R}^2$ be the graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \sin\left(\frac{2021}{x}\right), & \text{if } x \in [-1,1] \setminus \{0\}; \\ 2020, & \text{if } x = 0. \end{cases}$$

   Is the set $A \cup \{(0,y) \in \mathbb{R}^2 \ : \ -1 \le y \le 1\}$ measurable with respect to the Lebesgue measure on the plane? Justify your answer.

2. Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue integrable function and let $K \subset [0,1]$ be a measurable subset such that $\mu(K) > 0$ and $f(x) > 0$ for each $x \in K$ (here μ stands for the Lebesgue measure). Show that

$$\int_K f d\mu > 0.$$

3. Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue integrable function. Assume that

$$\int_0^x f d\mu = 0 \qquad \text{for each} \quad x \in [0,1].$$

   Show that $f(x) = 0$ for almost all $x \in [0,1]$.

4. Let $f : [a,b] \to \mathbb{R}$ be a C^1-function (that is f is differentiable and f' is continuous). Show that

$$\mathrm{Var}_a^b[f] = \int_a^b |f'(x)| dx$$

   (here $\mathrm{Var}_a^b[f]$ stands for the variation of f on $[a,b]$).

Good luck!

University of Texas at Dallas

Functional Analysis I

Qualifying Exam, August 2020

Problem 1. Let $\mathbb{E}$ be a Banach space and $C \subset \mathbb{E}$ a non-empty convex set.

(a) Show that if $x \in C$ and $y \in \text{int}(C)$ then for every $t \in (0,1)$ we have

$$tx + (1-t)y \in \text{int}(C);$$

(b) Show that if C is closed then C is also weakly closed (with respect to the weak topology $\sigma(\mathbb{E}, \mathbb{E}^*)$).

Problem 2. Consider the Euclidean space $\mathbb{E} := \mathbb{R}^n$, i.e. $\mathbb{E}$ is equipped with the norm

$$\|x\|_2 := \left[\sum_{k=1}^{n} x_k^2\right]^{\frac{1}{2}}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

Suppose C is an open bounded convex set such that $0 \in C$ and $C = -C$.

(a) Show that the function

$$\|x\| = \inf\{r > 0 : x \in rC\}, \quad x \in \mathbb{R}^n$$

is a norm on $\mathbb{R}^n$.

(b) Show directly that there exist constants C, $c > 0$ such that

$$\forall_{x \in \mathbb{E}} \;\; c\|x\|_2 \leq \|x\| \leq C\|x\|_2.$$

Problem 3. Let $\mathbb{E}$ be a reflexive Banach space and consider a continuous convex function $\varphi : \mathbb{E} \to \mathbb{R}$ such that $\lim_{\|x\|\to\infty} \varphi(x) = \infty$. Show that

$$\exists_{x_o \in \mathbb{E}} \quad \varphi(x_o) = \inf_{x\in\mathbb{E}} \varphi(x).$$

Problem 4. Consider the Banach space $\mathbb{E} := C([0,1],\mathbb{R})$ (equipped with the norm $\|\varphi\|_\infty := \sup_{t\in[0,1]} |\varphi(t)|$). Define the following linear operator $A : \mathbb{E} \to \mathbb{E}$ by

$$(A\varphi)(t) := \int_0^t \varphi(s)s^2 ds, \quad \varphi \in \mathbb{E},\ t \in [0,1].$$

(a) Show that the linear operator A is a bounded,

(b) Compute $\|A\|$.

Complex Analysis Qualifying Exam

Summer 2020

Friday, August 7, 2020

1. [**25 points**] **True** or **false** (Justification is needed):

(a) If $f(z)$ is analytic on a domain $D \subseteq C$, and γ is a closed curve in D, then $\int_\gamma f(z)dz = 0$. Is this true for any $f(z)$ and any D?

(b) If $f(z)$ is analytic on the unit disk $D = \{z : |z| < 1\}$, then there exists an $a \in D$, $a \neq 0$ such that $|f(a)| \geq |f(0)|$.

(c) Every analytic function $f(z)$ on a domain D has a power series expansion $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for each $z_0 \in D$ with a strictly positive radius of convergence.

2. [**25 points**]

(a) Given four distinct points z_1, z_2, z_3, z_4 in $\overline{C}$, their cross ratio, which is denoted (z_1, z_2, z_3, z_4), is defined to be the image of z_4 under the fractional linear transformation that sends z_1, z_2, z_3 to $\infty, 0, 1$, respectively. Prove that if ϕ is a fractional linear transformation then $(\phi(z_1), \phi(z_2), \phi(z_3), \phi(z_4)) = (z_1, z_2, z_3, z_4)$.

(b) Prove that the four distinct points z_1, z_2, z_3, z_4 of $\overline{C}$ lie on a (generalized) circle if and only if the cross ratio (z_1, z_2, z_3, z_4) is real.

(c) Compute the cross ratio $(2, -2, 2i, z)$ and use (b) to decide whether the points $\hat{z}_{1,2} = 1 \pm i\sqrt{3}$ and $\hat{z}_{3,4} = 2 \pm i$ lie on the circle $|z| = 2$.

3. [**25 points**]

(a) Let $H = \{z \in C \mid \operatorname{Im} z > 0\}$ be the upper half-plane. Let $f : H \to C$ be holomorphic and satisfy $|f(z)| \leq 1$ for $z \in H$ and $f(i) = 0$. Show that

$$|f(z)| \leq \left| \frac{z - i}{z + i} \right| \text{ for } z \in H.$$

(b) Let Ω be a bounded region, $a \in \Omega$ and $f : \Omega \to \Omega$ be a holomorphic map such that $f(a) = a$. Show that $|f'(a)| \leq 1$.

4. [**25 points**] Use the calculus of residues to evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{x^2 + 1}{(x^2 + 4)(x^2 + 9)} dx .$$

Verify all steps of the calculation.

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. There is one bonus problem. Show your work and justify all statements.

Problem 1 (25 points).

(a) (5 points) State the fundamental theorem of finitely-generated abelian groups.

(b) (20 points) Determine the structure (in the form of your statement of part (a)) of the finitely-generated abelian group generated by three elements x, y, z subject to the relations

$$\begin{aligned} 2x + 4y + 2z &= 0 \\ 12x + 6y + 4z &= 0 \\ 8x + 3y + 3z &= 0. \end{aligned}$$

Problem 2 (25 points). Let $\mathbb{Z}(G)$ be the group ring of G over the integers (that is, the ring of formal linear combinations of elements of G, with coefficients in $\mathbb{Z}$). Let I be a right ideal of $\mathbb{Z}(G)$ and define

$$G_I = \{g \in G | (1 - g) \in I\}.$$

Show that G_I is a subgroup of G, and that it is normal in G if I is a 2-sided ideal.

Problem 3 (25 points). Show that if H is a normal subgroup of G, then the center $Z(H)$ of H is a normal subgroup of G. Give an example of H and G such that H is a normal subgroup of G but $Z(H)$ is not a subgroup of $Z(G)$.

Problem 4 (25 points).

(a) (5 points) Let G be a finite group acting on a finite set X. State Burnside's lemma for the number of orbits $|X/G|$.

(b) (20 points) Let G be a finite group and let $g \in G$ and $h \in G$ be two randomly chosen elements (with replacement). Show that the probability that g and h commute is $k/|G|$, where k is the number of conjugacy classes in G. (*Hint: use part (a).*)

Bonus (10 points). Let p and q be primes with $p > q$ and $q \nmid (p-1)$. Show that all groups of order pq are cyclic.

ODE Qualifying Exam

August 2020

QE Id:

Problem 1) (15 pts) Find the principal matrix solution at $t_0 = 0$ for the following system.

$$\begin{cases} x_1' &= 2x_1 - x_2 \\ x_2' &= x_1 + 4x_2 \end{cases}$$

Problem 2) Solve the following initial value problems:

2a) (10 pts) $8txx' + 12x^2 = -4t$ where $x(1) = 1$

2b) (10 pts) $x' = 3t^2 + 2\sqrt{x - t^3}$ $x(0) = 4$

Problem 3) (15 pts) Find the solution to the following differential equation.

$$t^2x'' + tx' - 4x = \frac{12}{t} \qquad x(1) = 1 \quad x'(1) = 2$$

(Hint: $x = t^2$ is a homogeneous solution)

Problem 4) (15 pts) Prove or give a counterexample for the following statement:

Let $x' = f(t, x)$ with $x(t_0) = x_0$. If $f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, then there exists a unique solution $\phi(t)$ whose domain is $\mathbb{R}$.

[Here, $\mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ represents at least once differentiable functions whose domain is $\mathbb{R}^2$.]

Problem 5) (20 pts) Consider the following ODE: $x' = t - \ln x$

5a) Discuss the limit of the solution $x(t)$ when $t \to \infty$ for a given initial condition $x(t_0) = x_0 > 0$.

5b) Is there any solution where $|x(t)| \to \infty$ in finite time?

Problem 6a) (9 pts) Find the normalized eigenfunctions of the following problem.

$$y'' + \lambda y = 0 \qquad y'(0) = 0 \quad y(\pi) = 0$$

6b) (6 pts) By using part a, find a function $f(x)$ where the following equation has no solution.

$$y'' + \frac{9}{4}y = f(x) \qquad y'(0) = 0 \quad y(\pi) = 0$$

Qualifying Exam
Math 7329 August 2020
Topological and algebraic methods in nonlinear DEs

QE ID____________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Given a matrix

$$A = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & -d & c & 0 \\ -b & 0 & 0 & a \end{pmatrix}$$

and a vector $v = (1, 2, 3, 4)^t$, under which conditions with respect to the parameters a, b, c and d does the system

$$\dot{x} = Ax + \cos^{2020}(2t)v \qquad (x \in \mathbb{R}^4)$$

admit a periodic solution? Justify your answer.

2. Does the system

$$\begin{cases} \dot{x} = 4x^3 + 2xy^2 + 2xz^2 + (3y^2 + 4z^2 + 2020)\cos(2t) \\ \dot{y} = 2x^2y + 4y^3 + 2yz^2 + (4x^2 + 11z^2)\sin(2t) \\ \dot{z} = 2x^2z + 2zy^2 + 4z^3 + (8x^2 + 4y^2)\cos(2t) \end{cases}$$

admit a periodic solution? Justify your answer.

3. Let K_1 and K_2 be two compact sets in $\mathbb{R}^2$. Show that there exist $a, b, c \in \mathbb{R}$ such that:

$$\mu\{(x, y) \in K_1 \; : \; ax + by \geq c\} = \mu\{(x, y) \in K_1 \; : \; ax + by \leq c\}$$

and

$$\mu\{(x, y) \in K_2 \; : \; ax + by \geq c\} = \mu\{(x, y) \in K_2 \; : \; ax + by \leq c\}$$

(here "μ" stands for the Lebesgue measure on the plane).

4. Let $D \subset \mathbb{C}$ be the unit disc. Show that any two continuous paths γ_1 and γ_2 in $\overline{D}$ such that γ_1 connects 1 and -1 and γ_2 connects i with $-i$, must intersect in $\overline{D}$.

Good luck!

Combinatorics and Graph Theory QE

FALL 2020

QE ID:

INSTRUCTIONS. Write your QE ID above and solve all four problems. There is one bonus problem. Show your work and justify all statements.

Problem 1 (25 points).

(a) (5 points) State the Havel-Hakami Theorem, giving a condition for a degree sequence to be graphic.

Use the Havel-Hakami Theorem to determine which of the following degree sequences are graphic. If the degree sequence is graphic, draw a graph G with that degree sequence.

(b) (5 points) $d_1 = (3, 3, 3, 3, 2)$

(c) (5 points) $d_2 = (4, 3, 3, 2, 1)$

(d) (5 points) $d_3 = (4, 4, 3, 2, 1)$

(e) (5 points) $d_4 = (2, 2, 2, 1, 1)$

Problem 2 (25 points). Prove that if G is a planar graph, then $v - e + f = 2$, where v is the number of vertices of G, e is the number of edges, and f is the number of faces.

Problem 3 (25 points). Let F_n denote the number of ways to climb n steps using one or two steps at a time.

(a) (10 points) Show that $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(b) (15 points) Show that the ordinary generating function for the sequence $(F_n)_{n=0}^{\infty}$ is $F(x) = \frac{1}{1-x-x^2}$.

Problem 4 (25 points). Give a combinatorial proof that $\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. *(Non-combinatorial proofs will only receive partial credit.)*

Bonus (10 points). How many binary sequences $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ (with $\epsilon_i \in \{0, 1\}$) satisfy

$$\epsilon_1 \leq \epsilon_2 \geq \epsilon_3 \leq \epsilon_4 \geq \epsilon_5 \leq \cdots ?$$