Qualifying Exam
Math 6301 August 2018
Real Analysis I

QE ID____________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Give an example of a function $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable but is Lebesgue integrable.

2. Is the solution set to the equation

$$x^2 \sin\left(\frac{1}{x^{1998}+1}\right) + \exp\left(\cos\left(1 - x^{2018}\right)\right) + \sin\left(\exp\left(\frac{2019}{x^{2222}+2020}\right)\right) + \frac{2x^2}{2019 + x^6 + 2031} = 0$$

   a Lebesgue measurable subset of the real line? Justify your answer.

3. Does there exist a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(y)$ is measurable for any $y \in \mathbb{R}$? Justify your answer.

4. Let $f : [0,1] \to \mathbb{R}$ be a twisely differentiable function.

   (i) Is f'' a continuous function? Justify your answer.

   (ii) Is f'' a Lebesgue measurable function? Justify your answer.

5. Let $\{g_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of integrable functions convergent a.e. to g. Suppose, in addition, that g_n is non-negative a.e. for any $n \in \mathbb{N}$.

   (i) Is g integrable? Justify your answer

   (ii) Is the following statement true:

$$\int_{[0,1]} g_n e^{-g_n} d\mu \to \int_{[0,1]} g e^{-g} d\mu.$$

   Justify your answer.

Good luck!

University of Texas at Dallas

Functional Analysis I

Qualifying Exam, Summer 2018

Write your QE ID number (given to you by Angie)

Do NOT put your name, UTD ID, or any other identifying information on any of your answer sheets

Problem 1. Let $\mathbb{E}$ be a normed vector space and $A \subset \mathbb{E}$ a nonempty set. We denote by $\operatorname{conv}(A)$ the smallest convex set containing A.

(a) Show that $\operatorname{conv}(\overline{A}) \subset \overline{\operatorname{conv}(A)}$,

(b) Find an example of a set A such that $\operatorname{conv}(\overline{A}) \subsetneq \overline{\operatorname{conv}(A)}$,

(c) Show that if $\operatorname{int}(A) \neq \emptyset$ then $\operatorname{conv}(\overline{A}) \subset \overline{\operatorname{conv}(\operatorname{int}(A))}$.

Problem 2. Let $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ be Banach spaces.

(a) Show that if $A \in L(\mathbb{E}, \mathbb{F})$ then $A^* \in L(\mathbb{F}^*, \mathbb{E}^*)$ and $\|A\| = \|A^*\|$.

(b) Show that if A, $B \in L(\mathbb{E}, \mathbb{F})$ then $(A + B)^* = A^* + B^*$

(c) Show that if $A \in L(\mathbb{E}, \mathbb{F})$ and $B \in L(\mathbb{F}, \mathbb{G})$ then $(A \circ B)^* = B^* \circ A^*$;

(d) Show that if $\mathbb{E}$ and $\mathbb{F}$ are isomorphic, then $\mathbb{E}^*$ and $\mathbb{F}^*$ are also isomorphic.

Problem 3. Let $\mathbb{E}$ be a normed space and $x_o \in \mathbb{E}$. Show that

(a) there exists a functional $f_o \in \mathbb{E}^*$ such that $\|f_o\| = \|x_o\|$ and $\langle f_o, x_o \rangle = \|x_o\|^2$.

(b) for every $x \in \mathbb{E}$ we have

$$\|x\| := \sup_{\|f\| \leq 1} \langle f, x \rangle$$

(c) if for a linear functional $f : \mathbb{E} \to \mathbb{R}$ and an open set U containing x_o there exists a constant M such that $\langle f, x \rangle > M$ for all $x \in U$, then f is continuous.

Problem 4. Check that the functions $\varphi : \mathbb{R} \to (-\infty, +\infty]$ defined below are convex, l.s.c and determine the conjugate functions $\varphi* : \mathbb{R} \to (-\infty, +\infty]$.

(a) $\varphi(x) = \begin{cases} -\ln x & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0. \end{cases}$

(b) $\varphi(x) = \begin{cases} -\sqrt{1 - x^2} & \text{if } |x| \leq 1 \\ +\infty & \text{if } |x| > 1. \end{cases}$

(c) $\varphi(x) = \begin{cases} \frac{1}{2}|x|^2 & \text{if } |x| \leq 1 \\ |x| - \frac{1}{2} & \text{if } |x| > 1. \end{cases}$

Complex Analysis Qualifying Exam

August 10, 2018

1. [25 points] **True** or **false** (Justification is needed):
   a) If a power series $\sum_{n=0}^{\infty} a_n z^n$ has a finite non-zero radius of convergence, say r, then for every $w \in C$ with $|w| = r$, the series $\sum_{n=0}^{\infty} a_n w^n$ is divergent.
   b) The function $p(z) = z^5 + 3z^4 - 11z^3 + 4z + 2$ has 5 roots inside the unit disk $|z| < 1$.
   c) If $f(z)$ is nonconstant and entire then $\max_{|z|\le 1} |f(z)| < \max_{|z|\le 2} |f(z)|$.

2. [25 points] Let

$$f(z) = \begin{cases} \dfrac{x^3 y(y - i\,x)}{x^6 + y^2}, & \textit{if } z \neq 0, \\ 0, & \textit{if } z = 0. \end{cases}$$

   (a) Show that f is continuous at the origin.
   (b) Show that $\lim_{z\to 0} \dfrac{f(z) - f(0)}{z}$ exists along each ray, $\theta = \theta_0$, that all these limits are equal to 0, but that f is not differentiable at the origin.
   (c) Show that the Cauchy-Riemann equations hold at the origin.

3. [25 points]
   (a) Prove that if f is a holomorphic map from the unit disk $D = \{z : |z| < 1\}$ to itself with $f(0) = 0$, then $|f(z)| \le |z|$ for all $z \in D$.
   (b) Which of these f admit a point $a \neq 0$ in D with $|f(a)| = |a|$?
   (c) Find all analytic functions $h : D \to D$, such that $h(0) = 0$ and $h\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$.

4. [25 points] Given the function $f(z) = z^3 \cos\left(\dfrac{1}{z}\right)$.
   (a) Find in the domain $|z| > 0$ the Laurent series $\sum_{n=1}^{+\infty} \dfrac{b_n}{z^n} + \sum_{n=0}^{+\infty} a_n z^n$ of the function f. Indicate the coefficients a_n and b_n.
   (b) Indicate the isolated singularities of f in $\overline{C} = C \cup \{\infty\}$ and their types.
   (c) Find the value of the integral $\int_{|z|=1} f(z)\,dz$ and the residue of f at $z = \infty$.

Algebra Qualifying Exam

FALL 2018

QE ID:

INSTRUCTIONS. Write your name above and solve *exactly* 4 problems from the following list of 6 problems. Show your work, justify all statements, and clearly indicate which problem you are solving.

Problem 1. Prove that every group of order 312 is not simple.

Problem 2. Let G be a group with identity e. Suppose G has two elements x and y that satisfy the relations $xyx = y$ and $yxy = x$. Prove that $x^4 = y^4 = e$.

Problem 3. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

Problem 4. Let G be an infinite group in which every element has finite order and let S be a nonempty subset of G. Prove that S is a subgroup of G if and only if S is closed under the group multiplication.

Problem 5. For $x, y \in G$, write $[x, y] = x^{-1}y^{-1}xy$ for the *commutator* of x and y. Let

$$N = \langle [x, y] : x, y \in G \rangle$$

be the group generated by the commutators of all elements of G. Prove that $N \trianglelefteq G$ and that G/N is abelian.

Problem 6. Let R be a commutative ring with 1. We say $x \in R$ is *nilpotent* if $x^n = 0$ for some positive integer n.

1. Show that the set of nilpotent elements of R forms a subring of R.
2. If x is nilpotent, show that $1 + x$ is a unit.

Qualifying Exam: Ordinary Differential Equations I, August 2018

QE ID: ______________

THIS IS A CLOSED BOOK, CLOSED NOTES & NO CALCULATOR EXAM

Problems count 25 points each

Give clear and complete answers with full details in proofs

1. (a) Prove that if $f(x)$ satisfies the Lipschitz condition then $f(x)$ is continuous.
   (b) If $f'(x)$ is bounded then prove $f(x)$ satisfies the Lipschitz condition.

2. (a) State the needed condition(s) for the solution to an initial value ODE problem, $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, to exist and be unique, where $\mathbf{x}, \mathbf{f} \in R^n$ and $t \in R$.
   (b) Give a proof of your statement(s) in part (a).

3. Assume we are given the fundamental set of solutions $\Phi(t)$ for the differential equation $x' = A(t)x$, where prime indicates differentiation with respect to t.

   (a) Find the State Transformation matrix $\Psi(t, \tau)$ such that $\Psi'(t, \tau) = A(t)\Psi(t, \tau)$, with $\Psi(\tau, \tau) = E_n$, where E_n is the $n \times n$ identity matrix.

   (b) Prove that $x(t) = \Psi(t, \tau)\xi + \int_\tau^t \Psi(t, s)f(s)ds$ is the solution to the problem $x' = A(t)x + f(t)$ with $x(\tau) = \xi$.

4. Let $Lx = p_n(t)x^{(n)} + p_{n-1}x^{(n-1)} + \cdots + p_2(t)x'' + p_1(t)x' + p_0(t)x$ and $U_i x = \sum_{j=1}^{n}[\alpha_{ij}x^{(j-1)}(a) + \beta_{ij}x^{(j-1)}(b)]$ for $a \le t \le b$.
   (a) prove that the solution to the BVP $Lx = f$ with $U_i x = 0, \quad i = 1, 2, \cdots n$ is unique if $\lambda = 0$ is not an eigenvalue for the operator L.
   (b) What can you say about the uniqueness question for the case when $Lx = f$ but the boundary conditions are changed to $U_i x = \gamma_i \ne 0, \quad i = 1, 2, \cdots n$? Please prove the validity of your statement.

The University of Texas at Dallas

Qualifying Examination

MATH 6338: Delay Differential Equations

August 2018

QE ID: ______________________

Instructions:

1. Proved conclusions including theorems, lemmas and propositions can be cited without proof;
2. Use the space provided to write your solutions in this booklet;
3. Write your solutions neatly and legibly; Illegible or illogical work will be greatly discounted;
4. All questions are required.

Question	**Weight**	**Your Score**	**Comments**
1	25		
2	25		
3	25		
4	25		
Total:	100		

Question 1. Consider the equation

$$\dot{x}(t) = -\alpha x(t - \tau),$$

where $\tau > 0$ and $\alpha \in \mathbb{R}$.

i) Rescale the time variable t such that the equation has a time delay 1.

ii) Show that if $\alpha < 0$, the trivial equilibrium $x^* = 0$ is unstable.

iii) Show that if $0 < \alpha < \frac{\pi}{2\tau}$, the trivial equilibrium $x^* = 0$ is asymptotically stable.

Question 2. Consider the equation of $x(t) = (x_1(t),\, x_2(t)) \in \mathbb{R}^2$:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\dfrac{k}{m}x_1(t) - \dfrac{b}{m}x_2(t) - \dfrac{q}{m}x_2(t-\tau), \end{cases} \tag{1}$$

where m, k, b, q and τ are positive constants.

i) Denote by $x_t(\varphi)$ the solution with initial data $\varphi \in C([-\tau,\, 0]; \mathbb{R}^2)$. Show that if $q < b$, then for every $\varphi \in C([-\tau,\, 0]; \mathbb{R}^2)$, $x_t(\varphi)$, $t \geq 0$ is bounded.

ii) Let $q < b$. For every initial data $\varphi \in C([-\tau,\, 0]; \mathbb{R}^2)$, find the ω-limit set $\omega(\varphi)$. Justify your answer.

iii) Let $q < b$. Determine the existence of the limit $\lim_{t\to+\infty} x_t(\varphi)$. Justify your answer.

Question 3. Let $\tau > 0$ and $\alpha > 0$, and φ, ψ be differentiable functions with their segment functions $\varphi_t,\, \psi_t \in C([-\tau,\, 0]; \mathbb{R})$ for all $t \in [0,\, \alpha]$. Suppose that

$$\begin{aligned} \dot{\varphi}(t) &< f\left(t,\, \varphi(t),\, \sup_{t-\tau \le s \le t} \varphi(s)\right), \\ \dot{\psi}(t) &\ge f\left(t,\, \psi(t),\, \sup_{t-\tau \le s \le t} \psi(s)\right), \end{aligned} \tag{2}$$

where $(0,\, \alpha) \times \mathbb{R}^2 \ni (t,\, u,\, v) \mapsto f(t,\, u,\, v) \in \mathbb{R}$ is continuous and strictly increasing in v. Show that if $\varphi(s) < \psi(s)$ for $s \in [-\tau,\, 0]$, then

$$\varphi(t) < \psi(t)$$

for every $t \in (0,\, \alpha)$.

Question 4. Consider the Nicholson's blowfly equation

$$N'(t) = -\delta N(t) + pN(t-r)e^{-qN(t-r)}, \tag{3}$$

where δ, r, p, and q are positive parameters. Denote by $C_+ = \{\varphi \in C([-r,\,0];\mathbb{R}) : \varphi \geq 0\}$.

i) Show that $V : C_+ \to \mathbb{R}$ defined by

$$V(\varphi) = \varphi(0) + \int_{-r}^{0} p\,\varphi(s)e^{-q\varphi(s)}\mathrm{d}s,$$

is a Lyapunov function on some subset G of C_+.

ii) Show that if $\frac{p}{\delta} \leq 1$, then ω-limit set $\omega(\varphi) = \{\hat{0}\}$, where $\varphi \in C_+$, $\hat{0}$ denotes the zero constant function in C_+.

iii) Let $x_t(\varphi)$, $t \geq 0$ be a solution with initial data $\varphi \in C_+$. Does the limit $\lim_{t\to+\infty} x_t(\varphi)$ exist? Do we have to assume $\frac{p}{\delta} \leq 1$ for the existence of this limit? Explain your answer.

Numerical Linear Algebra Qualifying Exam August 2018

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

(1) We want to solve the linear system $Ax = b$ given by

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(a) Apply two iterations of the Gauss-Seidel algorithm to this system. Use $\vec{x}^0 = (0,0)^t$ as your initial solution guess.

(b) Will the Gauss-Seidel method converge to the true solution for this problem? **Why or why not?**

(c) What would be a reason to use an iterative technique like Gauss-Seidel rather than solving the linear system directly using a technique like Gaussian Elimination?

(d) Show that the computational cost of forward elimination in Gaussian elimination is $O(n^3)$.

(2a) Give conditions on a matrix A such that the following techniques can be used to solve the least squares problem $Ax = b$:

(i) the QR factorization; (ii) setting up and solving the normal equations; (iii) using the pseudoinverse from the SVD.

(b) what is the minimum length least squares solution x^+ to $Ax = b$ if

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}?$$

(3) Consider the explicitly-shifted QR iteration

$$A_k - \alpha_k I = Q_k R_k$$

$$A_{k+1} = R_k Q_k + \alpha_k I$$

where A_0 is a given complex $n \times n$ matrix, R_k is an upper triangular matrix, and Q_k is a unitary matrix for $k = 0, 1, \ldots$.

(a) Prove that A_k is similar to A_0 for all k.

(b) Discuss why the algorithm is usually applied to a matrix A_0 which has already been reduced to upper Hessenberg form.

(c) Give a strategy for determining the shift α_k for upper Hessenberg matrices, and state the rate of convergence of the eigenvalue estimate to the true eigenvalue.

(d) Suppose z is the column vector $(1, 0, \ldots, 0)^t$, $\sigma = ||x||_2$, and $v = x + \sigma z$ is the vector used to create the Householder transform H. Prove that $Hx = -\sigma z = (-\sigma, 0, \ldots, 0)^t$.

(4) Define the matrix A_n of order n by

$$A_n = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & \cdots & -1 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

(a) Find the inverse of A_n explicitly. (Hint: Find the inverse of A_4 or A_5, then use this result to "guess" the inverse of A_n in general.)

(b) Calculate $\text{cond}(A_n)$.

(c) With $b = [-n+2, -n+3, \ldots, -1, 0, 1]^t$ the solution of $A_n x = b$ is $x = [1, 1, \ldots, 1]^t$. Perturb b to $\hat{b} = b + [0, \ldots, 0, \epsilon]^t$. Solve for $\hat{x}$ in $A_n \hat{x} = \hat{b}$ for the case when $n = 4$. Very carefully explain how this result agrees/disagrees with the theory concerning how the solution of a linear system is affected by perturbations in the right hand side vector.

MATH 6310 (SUMMER 2018) – QUALIFYING EXAM

QE ID:

There are 5 problems. Each problem is worth 20 points. The total score is 100 points.

Show all your work to get full credits.

Problem 1. The real projective n-space $\mathbb{R}P^n$ is defined as the quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ by the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$. Find a cell decomposition of $\mathbb{R}P^n$.

Problem 2. Compute the fundamental group of the quotient space of the unit sphere $S^2 \subset \mathbb{R}^3$ obtained by identifying the points $(1, 0, 0)$ and $(0, 1, 0)$.

Problem 3. Let X be the topological space obtained from a triangle by identifying its three vertices to a single point. Compute the (simplicial) homology groups of X.

Problem 4. Prove the Brouwer fixed point theorem: Every continuous map $f : D^n \to D^n$ has a fixed point, that is, a point x with $f(x) = x$.

Problem 5. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the (cellular) homology groups of X.