Qualifying Exam
Math 6301 Summer 2017
Real Analysis

Name______________________________

Instructions: *Please solve the following problems. Work on your own and do not discuss these problems with your classmates or anyone else.*

1. Show that a set $A \subseteq E$, where E is given by

$$E := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\},$$

is Lebesgue measurable iff for every $\epsilon > 0$, there is an open set $G \subseteq E$ and a closed $F \subseteq E$, such that

$$F \subseteq A \subseteq G$$

and

$$\mu\,(G \backslash F) < \epsilon$$

2. Consider a sequence of functions

$$f_n(x) = \frac{n \sin(x)}{1 + n^2 \sin^2(x)}, \quad (x \in [0, \pi].$$

(i) Does this sequence converge pointwisely? Justify your answer.

(ii) Does this sequence converge uniformly? Justify your answer.

(iii) Take $\delta > 0$. Find *explicitly* a set $E_\delta \subset [0, \pi]$ such that the sequence $\{f_n\}$ converges uniformly on E_δ and $\mu\big([0, \pi] \setminus E_\delta\big) < \delta$.

3. (a) Let

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset [a, b]$$

be a sequence of measurable sets and let $f : [a, b] \to \mathbb{R}$ be a non-negative integrable function. Show that

$$\lim_{n\to\infty} \int_{A_n} f(x) d\mu = \int_A f(x) d\mu, \quad \text{where} \quad A := \bigcup_{n=1}^{\infty} A_n.$$

(b) Is the above statement true without the hypothesis "f is non-negative"? Justify your answer.

4. Compute $\int_{[0,\pi/2]} f d\mu$, where

$$f(x) = \begin{cases} \sin^5(x), & \text{if} \;\; \cos(\mathrm{x}) \;\; \text{is} \;\; \text{rational;} \\ \cos^6(x), & \text{if} \;\; \cos(\mathrm{x}) \;\; \text{is} \;\; \text{irrational} \end{cases}$$

University of Texas at Dallas

Functional Analysis I

Qualifying Exam, Summer 2017

Instructions

(a) Attempt to solve all four questions, however only the best three of your solutions will be considered.

(b) Include all the related to the problems definitions and state all the results that you are using in your solutions.

(c) Please write clearly and use your notation carefully, so we can avoid unnecessary misunderstandings.

Problem 1. (a)–(e)

(a) Let $\mathbb{E}$ be a normed space and $\varphi : \mathbb{E} \to (-\infty, \infty]$ a given function.
(i) Give a definition of the *conjugate function* $\varphi^* : \mathbb{E}^* \to (-\infty, \infty]$.
(ii) List all the properties of the conjugate function φ^*.

(b) For a linear subspace L of the normed space $\mathbb{E}$ compute ψ^*, where $\psi(x) = I_L(x)$ is the indicatrice function of L.

(c) Let $\mathbb{E}$ be a normed space, $x_o \in \mathbb{E}$ a fixed vector and $\varphi(x) = \|x - x_o\|$, $x \in \mathbb{E}$. Compute φ^*.

(d) State Fenchel-Rockaffellar Theorem.

(e) Let $\mathbb{E}$ be a normed space. Suppose that $L \subset \mathbb{E}$ is a linear subspace and let $x_o \in X$ be such that $\operatorname{dist}(x_o, L) = d > 0$. Show that there exists $f_o \in \mathbb{E}^*$ such that $\langle f_o, x_o \rangle = d$, $\|f_o\| = 1$ and

$$\forall_{x \in L} \quad \langle f_o, x \rangle = 0.$$

Problem 2. Let $\mathbb{E}$ be a Banach space and $T : \mathbb{E} \to \mathbb{E}^*$ a linear operator satisfying

$$\forall_{x,y\in\mathbb{E}} \quad \langle Tx, y\rangle = \langle Ty, x\rangle. \tag{1}$$

Prove that T is a bounded operator.

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Problem 3. Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces and $A : \mathbb{E} \to \mathbb{F}$ a bounded linear operator.

(a) Give a definition of the operator norm $\|A\|$;

(b) Show that $\|\cdot\|$ is indeed a norm on $L(\mathbb{E}, \mathbb{F})$ (check all the required conditions);

(c) Give the definition 0f the adjoint operator A^*;

(d) Show that if $A : \mathbb{E} \to \mathbb{F}$ is an isomorphic, then A^* is also an isomorphism.

Problem 4. Assume that $V = \mathbb{R}^n$ and let $\|\cdot\| : V \to \mathbb{R}$ be an arbitrary norm on V. Put $\varphi(x) := \|x\|$ for $x \in V$.

(a) Show that the function φ is convex.

(b) Show that

$$\exists_{c,d>0}\, \forall_{x\in V} \quad \|x\|_\infty = 1 \quad \Rightarrow \quad c \le \varphi(x) \le d,$$

where

$$\|x\|_\infty := \max_{1\le k\le n} |x_k|, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

(c) Show that the norms $\| \cdot \|$ and $\| \cdot \|_\infty$ are equivalent.

Complex Analysis Qualifying Exam

Summer 2017

August 11, 2017

1. The differentiability is considered in the complex sense.

(a) Determine, using the definition of differentiability, whether $f : C \to C$ given by $f(z) = |z - z_0|$ is differentiable or not differentiable at $z = z_0$. Prove your answer.

(b) Let $a \in R$ be fixed and $f(z) = z - \bar{z} + a \cdot \bar{z}^2$. Determine the set of points $z \in C$ at which $f(z)$ is differentiable. Is $f(z)$ analytic at these points?

2. Find all fractional-linear transformations that map the lower half-plane onto the unit disk.

3. Does there exist an analytic function $f : D \to D$ with $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$, where D is the open unit disk?

4. Use the calculus of residues to evaluate the improper integral

$$\int_0^\infty \frac{\cos x}{(x^2+1)^3}\, dx.$$

Verify all steps of the calculation.

Abstract Algebra Qualifying Exam

August 9, 2017

Name_________________________

Instructions. *Please solve any four problems from the list of the following problems (show all your work).*

1. Let D_6 be the dihedral group of symmetries of a regular hexagon.

a) Find all subgroups of order 4 in D_6 and the center $Z(D_6)$ of D_6.

b) Find all subgroups of D_6 that are isomorphic to the dihedral group D_3 of symmetries of an equilateral triangle.

c) Give an example of a Sylow 3-subgroup of D_6.

2. Let G be a group, $Z(G)$ denotes the center of G, and $\mathsf{Inn}(G)$ be the group of all inner automorphisms of G.

a) Suppose that $Z \leq Z(G)$ and G/Z is cyclic. Show that G is abelian.

b) Find the group $\mathsf{Inn}(D_4)$ of all inner automorphisms of D_4, where D_4 is the dihedral group of symmetries of a square.

3. Let G be a group that acts on the set Ω. Prove that, if $a, b \in \Omega$ and $b = g \cdot a$, for some $g \in G$, then

$$G_b = gG_a g^{-1},$$

where G_x is the stabilizer of x. Deduce that if G acts transitively on Ω, then the kernel of the action is $\bigcap_{g \in G} gG_a g^{-1}$.

4. Let G be a group and $x, y \in G$. Define the commutator of x and y by $[x, y] = x^{-1}y^{-1}xy$. Let $[G, G] \leq G$ be a subgroup of G generated by the set $\{[x, y] \mid x, y \in G\}$.

a) Show that $[G, G]$ is a characteristic subgroup of G, i.e. $[G, G]$ char G.

b) Show that if H is a normal subgroup of G (i.e.$H \trianglelefteq G$) and K char H then $K \trianglelefteq G$.

5. Let G be a group. Show that:

a) Show that if $H \leq G$ and $|G : H| = 2$ then $H \trianglelefteq G$.

b) If the order of G is 56 then G is not simple.

6. Prove the following statements:

a) Let M and N be normal subgroups of a group G with $M \cap N = \{1\}$. Then $mn = nm$ for all $m \in M$ and $n \in N$.

b) Prove that no group of order 48 is simple.

Name: ______________

Qualifying Exam: Ordinary Differential Equations I, August 2017

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Problems count 25 points each
Give clear and complete answers with full details in proofs

1. Consider the initial value problem $x' = x^{1/5}$ and $x(0) = x_0$ for $t \geq 0$.

   a) Is the solution to the above IVP unique if $x_0 = 1$? Clearly justify/prove your answer.

   b) Is the solution to the above IVP unique if $x_0 = 0$? Clearly justify/prove your answer.

2. Consider the differential operator L, defined by the operation $Lu = i\frac{du}{dt}$, acting on differentiable functions u defined over $[0, 2]$ with the condition that $u(2) = \alpha u(0)$, where $i = \sqrt{-1}$ and α is a complex number. Find a general condition on α that will make the differential operator L to be self adjoint.

3. Consider the boundary value problem on $[0, \pi]$ for the equation

$$x'' + \lambda x = 0,$$

   with $x'(0) = 0$ and $x(\pi) = 0$.

   a. Are there any eigenvalues which are not real?

   b. Find the eigenvalues

   c. Find the corresponding eigenfunctions.

4. Define the successive approximate solutions ϕ_k of $x' = f(t, x)$ with $x(\tau) = \xi$? If $f\epsilon(C, Lip)$ on R, then give a detailed proof that ϕ_k's converge uniformly to a unique solution ϕ of the above IVP.

MATH 6319 - Ph. D Qualifying Examinations
Summer 2017
V. Ramakrishna

CHOICE: Do any **4** of the Qs below. Each **Q** is worth 25 points.

- **Q1** Compute in closed form the eigenvalues of the matrix $A = (i + j)$. Explain your work.

- **Q2** State carefully the term recurrence relations satisfied by orthonormal polynomials. Reformulate this recurrence in terms of matrices and explain why this implies that the zeroes of such polynomials are real and simple.

- **Q3**

State and prove Fischer's inequality for positive semidefinite matrices.

- **Q4** State and prove the Crabtree-Haynsowrth formula. You may assume a certain formula (which, however, must be stated precisely at the point of usage) regarding Schur complements in a product of 3 matrices.

Q5 Show that the Fourier transform of a Gaussian is another Gaussian, by setting up a differential equation for the Fourier tarnsform.

Q6 Let f and g equal the characteristic function of $[0, 1]$. Compute in closed form their convolution.